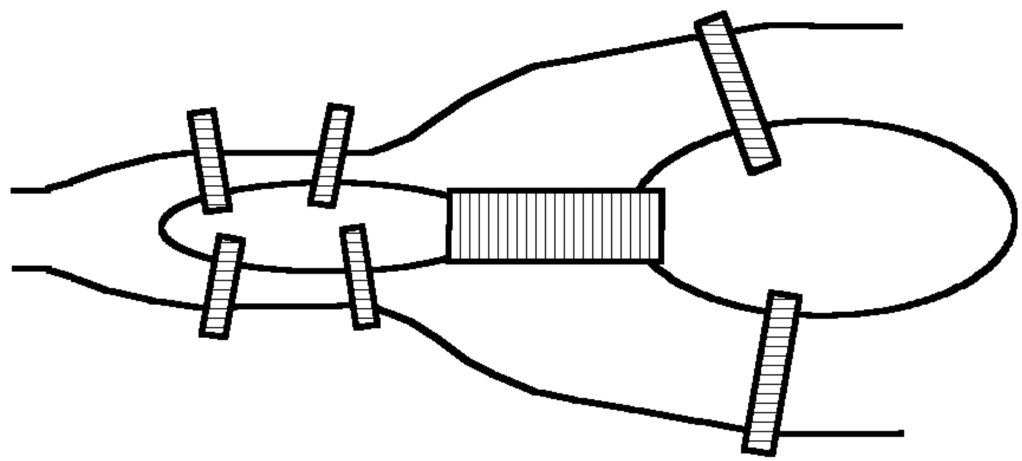
Networks

- People have been thinking about network problems for a long time
- Koenigsberg Bridge problem (Euler, 1736)



The Konigsberg Bridge Problem

- Can you cross all 7 bridges exactly once on a walk?

Types of Network Flow Problems (Winston)

- Transportation problem (Hitchcock, 1941)
  - Given a set of sources, destinations
  - Have source capacities, destination demands
  - Know shipping cost/unit for each source-destination pair
  - Minimize total cost of meeting demands at destinations
- Assignment Problem
  - Restricted transportation problem
  - Each source can supply 1 object
  - Each destination demands 1 object
- Transshipment Problem
  - Transportation problem with intermediate (transshipment) points
  - Generalizes both transportation and assignment problem

Other Network Models in Winston

- Shortest path problem (Dijkstra1959)
  - Find the shortest route between an origin and a destination
  - Algorithms range simple “greedy” algorithm to very complex
  - Special case of the transshipment problem
- Maximum flow problem
  - Maximize flows from a single source to a single destination
  - Important dual result: finds the minimum “cut” in a network
- Minimum spanning tree
  - Objective is to connect all nodes in a network
  - Want to minimize the total length of the connections
- Project management (PERT-CPM)
  - Find time to complete a linked set of tasks
  - Is actually a “longest path” problem

Some Network Models *Not* in Winston

- Circulation problem
  - Transshipment problem with no source or demand nodes
  - Example: airline routing schedules
- Generalized flow problem
  - Some of the material flowing on the network is lost in transmission
  - Example: electrical power transmission, water distribution
- Multicommodity flow problem
  - More than one type of material is flowing on the network
  - Different materials consume network capacity, but may have different transmission costs
- Network interdiction problem
  - Bad guys are moving on a network; good guys try to stop them
  - Each side has to choose what paths to use or interdict

Network Models *Not* in Winston (cont'd)

- Network reliability problem
  - Find the maximum reliability set of routes in network
  - In certain cases, can be recast as an MCNFP
- Network models with "side constraints"
  - Knapsack problem: optimize the "goodness" of a set of objects that must fit into a finite-sized "knapsack"
  - Traveling Salesman Problem: find the minimum-cost tour among a set of destinations, but only visit each destination once
  - Various vehicle routing problems

- Network optimization literature is gigantic
  - Seminal text is Ahuja, Magnanti, and Orlin (1993); 846 pages!
  - Huge number of applications
  - Inspiration for much of the research into algorithmic efficiency

Concentration in This Course

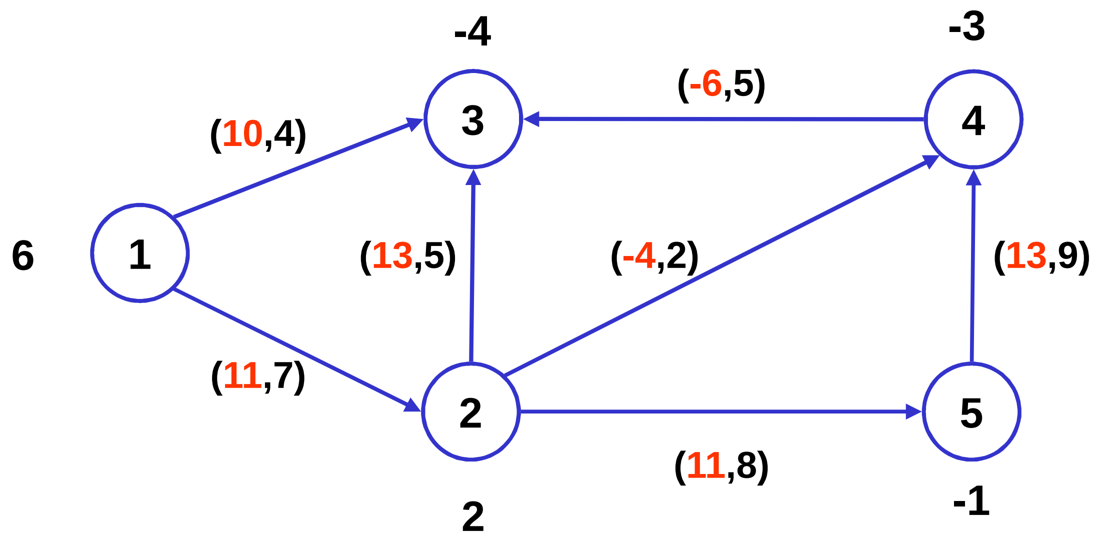
- I will emphasize the transshipment problem
- Otherwise known as the minimum-cost network flow problem (MCNFP)
- Reasons:
  - Transportation, assignment, max flow, shortest path problems are all *cases* of MCNFP
  - All commercial software implements a modified simplex method based on MCNFP
  - To exploit this capability, you have to formulate in terms of MCNFP
  - Specialized transportation and assignment problem algorithms are largely unnecessary (e.g., stepping-stone and Hungarian methods)

Network Jargon

- To model a problem as a network, we need some terminology
- A graph ***G*** consists of:
  - A set of nodes (***N***)
  - A set of arcs (***A***), which connect the nodes
  - So, ***G*** = (***N,A***) specifies the "topology" of the network
- Graph characteristics
  - Let ***i,j*** be indices for the nodes
  - Then the pair (***i,j***) identifies an arc
  - This notation allows us to define things such as:
    - Unit transportation costs on an arc (C_{ij})
    - Supplies at each source node (S_i)
    - Demands at each demand node (D_j)

From Network to Optimization Model

- Consider the transshipment network below:



- What does all this mean?
  - Numbers in circles are node labels
  - Numbers next to nodes are *supplies* (>0) and *demands* (<0)
  - Numbers on arcs are (cost/unit to ship, max flow on arc)
- Assume we want to minimize total cost

Optimization Model (Winston format)

- Let $\boldsymbol{x_{ij}}$ be the flow on arc (***i,j***)
- Then, the model is:

$$
\begin{aligned}
&\min\ z = 11x_{12} + 10x_{13} + 13x_{23} - 4x_{24} + 11x_{25} - 6x_{43} + 13x_{54} \\
&\text{subject to} \\
&x_{12} + x_{13} = 6 \quad \text{(node 1 balance)} \\
&-x_{12} + x_{23} + x_{24} + x_{25} = 2 \quad \text{(node 2 balance)} \\
&-x_{13} - x_{23} - x_{43} = -4 \quad \text{(node 3 balance)} \\
&-x_{24} + x_{43} - x_{54} = -3 \quad \text{(node 4 balance)} \\
&-x_{25} + x_{54} = -1 \quad \text{(node 5 balance)} \\
&\text{all } x_{ij} \geq 0 \\
&x_{12} \leq 7,\ x_{13} \leq 4,\ x_{23} \leq 5,\ x_{24} \leq 2,\ x_{25} \leq 8,\ x_{43} \leq 5,\ x_{54} \leq 9
\end{aligned}
$$

Some Key Points

- FLOW MUST BALANCE!
  - For pure supply nodes, flow out must equal supply
  - For pure demand nodes, flow in must equal demand
  - For transshipment nodes, flow out must equal flow in
  - Note the example has nodes that demand and transship
- What do we do if supply is unequal to demand?
  - Create dummy node to absorb (ship) excess supply (demand)
  - Create costs that make sense in the model
- Arc costs can be negative
- Can also force flow on arcs by putting in lower bounds
- Sign conventions
  - Flow out is positive
  - Flow in is negative

Optimization Model (Algebraic format)

- Indices
  - ***i, j*** = nodes {1,2,3,4,5}
- Subsets
  - ***ARCS(i,j)*** = connections between nodes
- Data
  - C_{ij} = cost per unit to ship on arc ***i,j***
  - L_{ij} = lower bound on flow on arc ***i,j***
  - U_{ij} = upper bound on flow on arc ***i,j***
  - SD_i = supply or demand at node ***i***
- Variables
  - x_{ij} = flow on arc ***i,j***

Algebraic Format (cont'd)

- So, the general MCNFP is:

$$\min z = \sum_{i,j \in ARCS(i,j)} C_{ij} * x_{ij}$$

subject to

FLOW IN

$$\left[\sum_{j \in ARCS(i,j)} x_{ij} \right] - \left[\sum_{j \in ARCS(j,i)} x_{ji} \right] = SD_i \text{ for all nodes } i$$

FLOW OUT

$$L_{ij} \le x_{ij} \le U_{ij} \text{ for all } ARCS(i,j)$$

- One MPL program can represent any MCNFP!
- So, are we done yet?

Answer is NO

- The trick with networks is to translate the problem into a network structure
- Many things that don't appear to be networks are
  - See Winston, pp. 366-368 on the inventory problem
  - Many, many other models can be represented by a network
- So, the challenge is to:
  - Determine if the problem can be represented by a network
  - If so, come up with the nodes, arcs, costs, and capacities
- **Why do we want to do this?**
  - **Speed:** network codes are much faster than normal LP simplex
  - **Integrality:** if a problem can be represented by an MCNFP, it will have integral answers (if the supplies, demands and bounds are integer)

Common MCNFP Models - Transportation

- Transportation problem
  - This is an MCNFP with no transshipment nodes
  - Nodes are divided into two disjoint sets, SOURCES and SINKS
- So, the formulation becomes:

$$\min z = \sum_{i,j \in ARCS(i,j)} C_{ij} * x_{ij}$$

subject to

$$\sum_{j \in ARCS(i,j)} x_{ij} = SUPPLY_i \text{ for all nodes } i \in SOURCES$$

$$\sum_{i \in ARCS(i,j)} x_{ij} = DEMAND_j \text{ for all nodes } j \in SINKS$$

$$0 \le x_{ij} \le U_{ij} \text{ for all } ARCS(i,j)$$

- Winston shows a special algorithm for this, but it's unnecessary

Common MCNFP Models - Assignment

- A very simple model (too simple to be of much use)
  - We are matching demands to supplies
  - Each demand (supply) node demands (supplies) 1 unit
  - Number of supply and demand nodes are equal
- So, this model is:

$$
\begin{aligned}
&\min z = \sum_{i,j \in ARCS(i,j)} C_{ij} * x_{ij} \\
&\text{subject to} \\
&\sum_{j \in ARCS(i,j)} x_{ij} = 1 \text{ for all nodes } i \in SOURCES \\
&\sum_{i \in ARCS(i,j)} x_{ij} = 1 \text{ for all nodes } j \in SINKS \\
&x_{ij} \in \{0,1\} \text{ for all } ARCS(i,j)
\end{aligned}
$$

- Again, Winston shows a special (Hungarian) algorithm for this - it's not needed

Common MCNFP Problems - Max Flow

- This is useful, but is a twist on the MCNFP formulation
  - We are maximizing flow from a single source (s) to a single destination (d)
  - This flow, however, is a variable, ***v***
- This model is:

$$\max\ z = v$$

$$\text{subject to}$$

$$\left[\sum_{j \in ARCS(i,j)} x_{ij}\right] - \left[\sum_{j \in ARCS(j,i)} x_{ji}\right] = \begin{cases} v & \text{for } i = s \\ 0 & \text{for all nodes } \ i \neq s, i \neq d \\ -v & \text{for } \ i = d \end{cases}$$

$$0 \leq x_{ij} \leq U_{ij} \ \text{ for all } ARCS(i,j)$$

- Note here that the variable *v* represents a "return arc" from *d* to *s*

Shortest (Longest) Path Problem

- This can also be represented by an MCNFP
  - Costs are arc lengths
  - Flow 1 unit from the origin s to the destination d
  - Minimize (maximize) the sum of the arcs used
- Formulation:

$$\min \left(\text{or max}\right)\ z = \sum_{i,j \in ARCS(i,j)} C_{ij} * x_{ij}$$

subject to

$$\left[\sum_{j \in ARCS(i,j)} x_{ij}\right] - \left[\sum_{j \in ARCS(j,i)} x_{ji}\right] = \left\{\begin{array}{c} 1, i = s \\ 0, i \neq s \text{ and } i \neq d \\ -1, i = d \end{array}\right\}$$

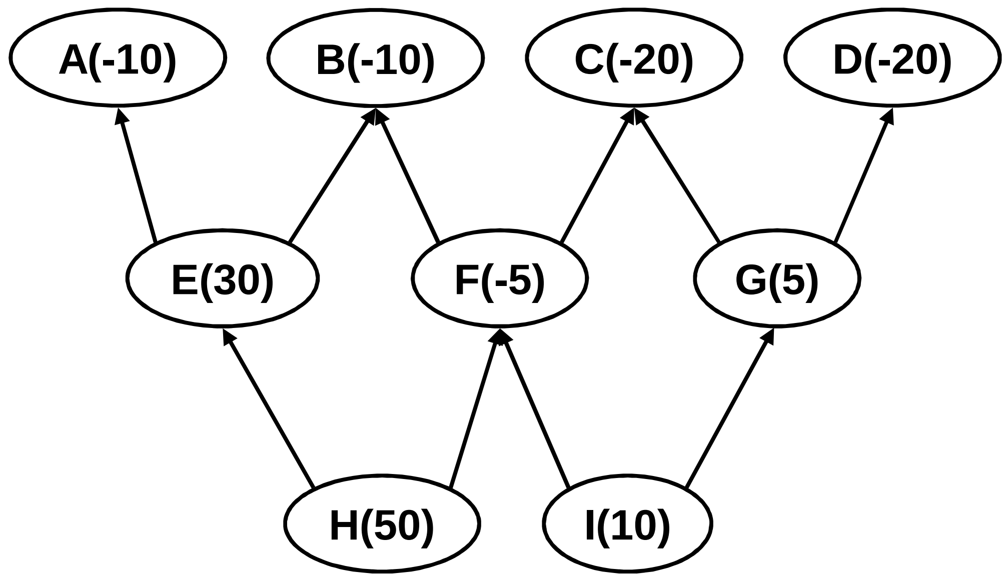
$$0 \leq x_{ij} \leq 1 \text{ for all } ARCS(i,j)$$

- NOTE: lots of simple algorithms (e.g., Dijkstra) available for this problem

Converting a Problem to a Network

- Many hard problems become easy if you can convert them to a network
- Example: open-pit mining problem
  - An open pit mine can be represented as a set of blocks
  - Each block *i* has a net profit w_i if you choose to extract it
  - But, you have to remove the blocks above it to get to it
- This problem is called a “maximum weight closure”
  - The graph is a set of nodes with weights
  - The arcs show dependencies among nodes
  - A closure is a set of nodes with no outgoing arcs
  - The objective is to find the closure with maximum weight

Example Closure Problem



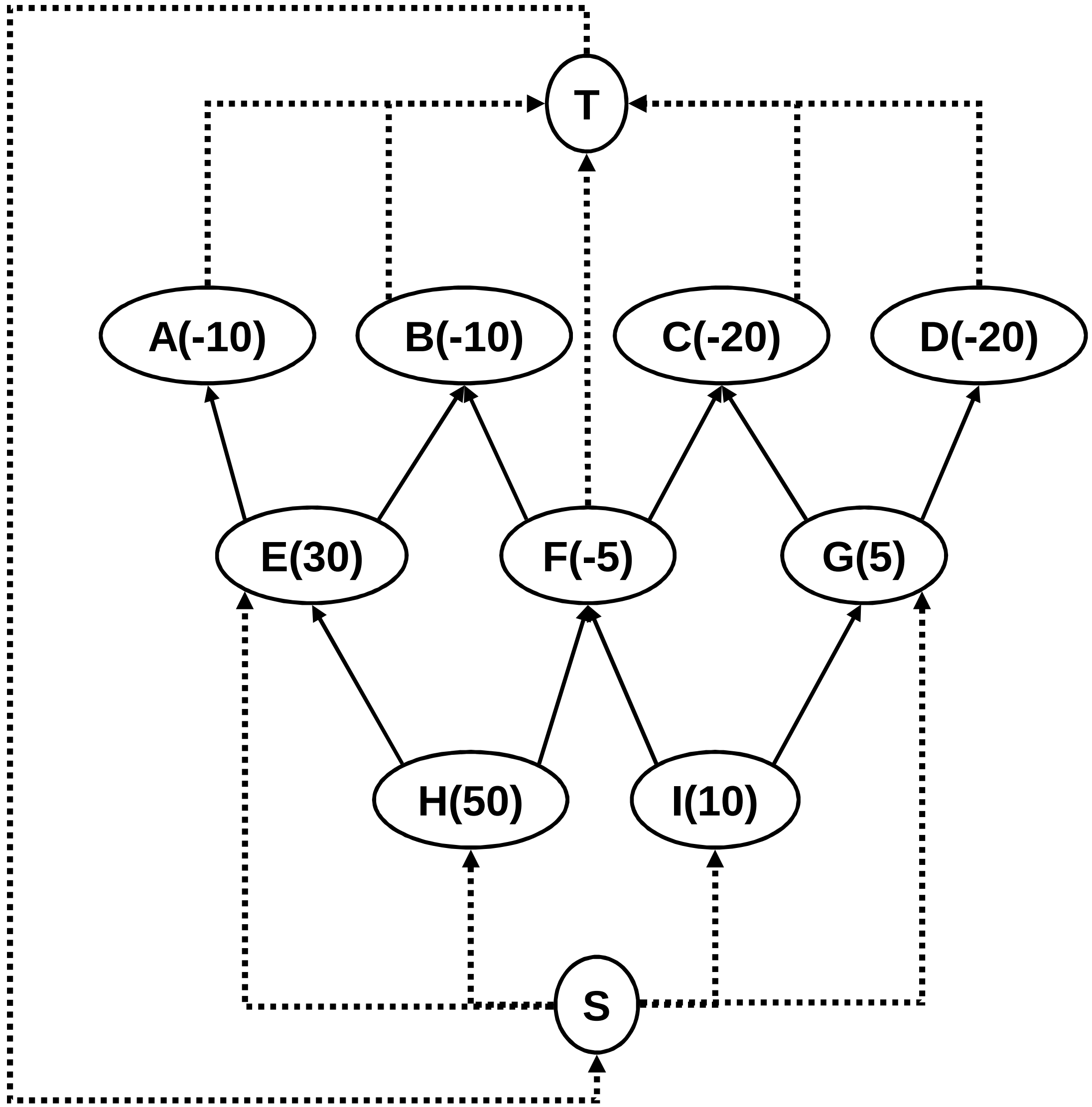
Number in parentheses is payoff for choosing that node

Example: A, B, and E is a closure, with total weight = 10

Converting a Closure to a Max Flow Problem

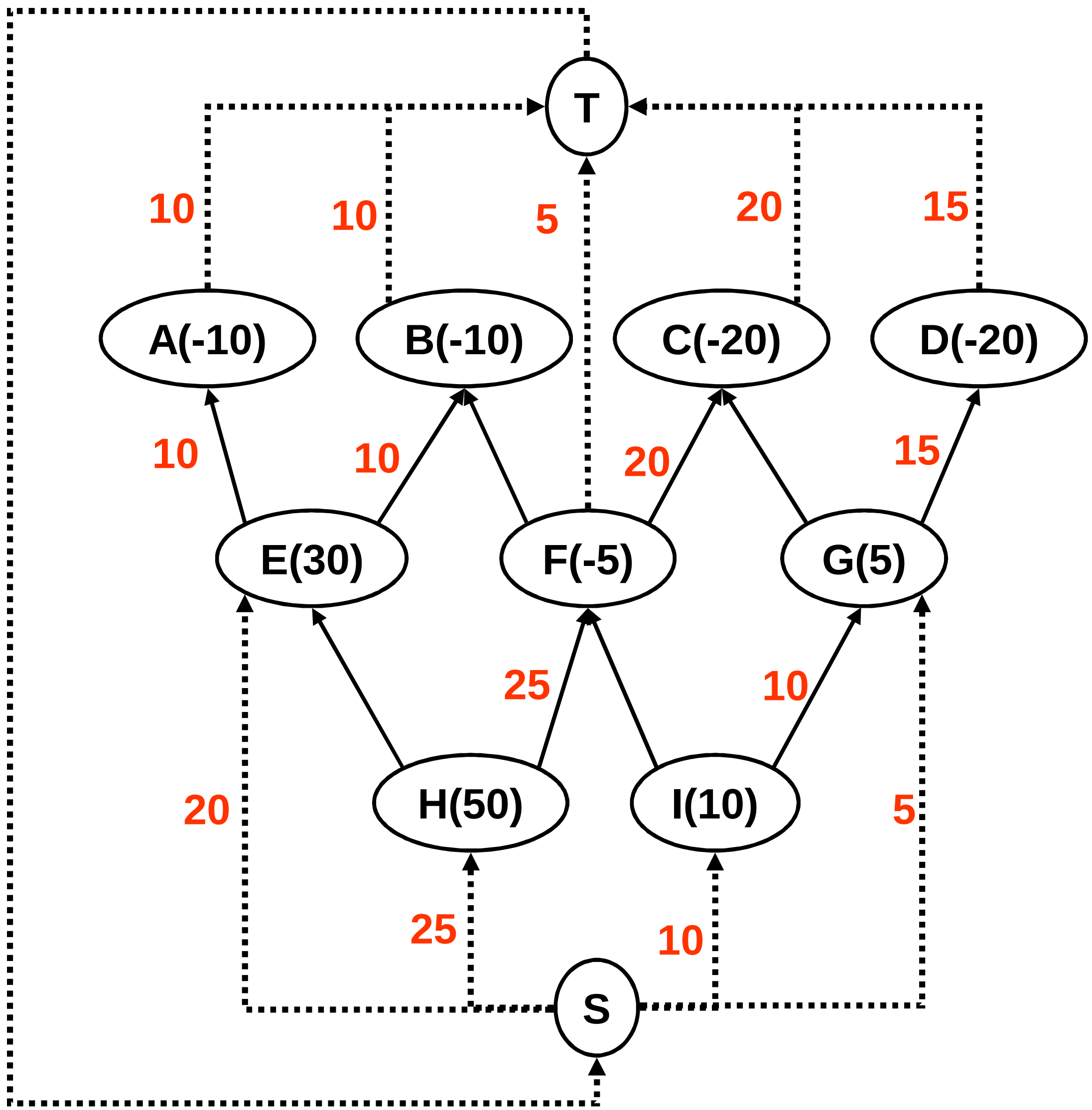
- Add two new nodes, *s* and *t*
- Connect all nodes with positive payoffs with arcs *from* node *s*
- Connect all nodes with negative payoffs arcs *to* node *t*
- Make the upper bound on all new arcs the absolute value of the weight of the node
- Make the upper bound on the original arcs infinite
- Solve a maximum flow problem with this network

The Maximum Weight Closure as a Max Flow



Upper bound on flow for all new arcs is the absolute value of the node they connect to (except for T to S)

Getting the Answer



Objective function value: 60

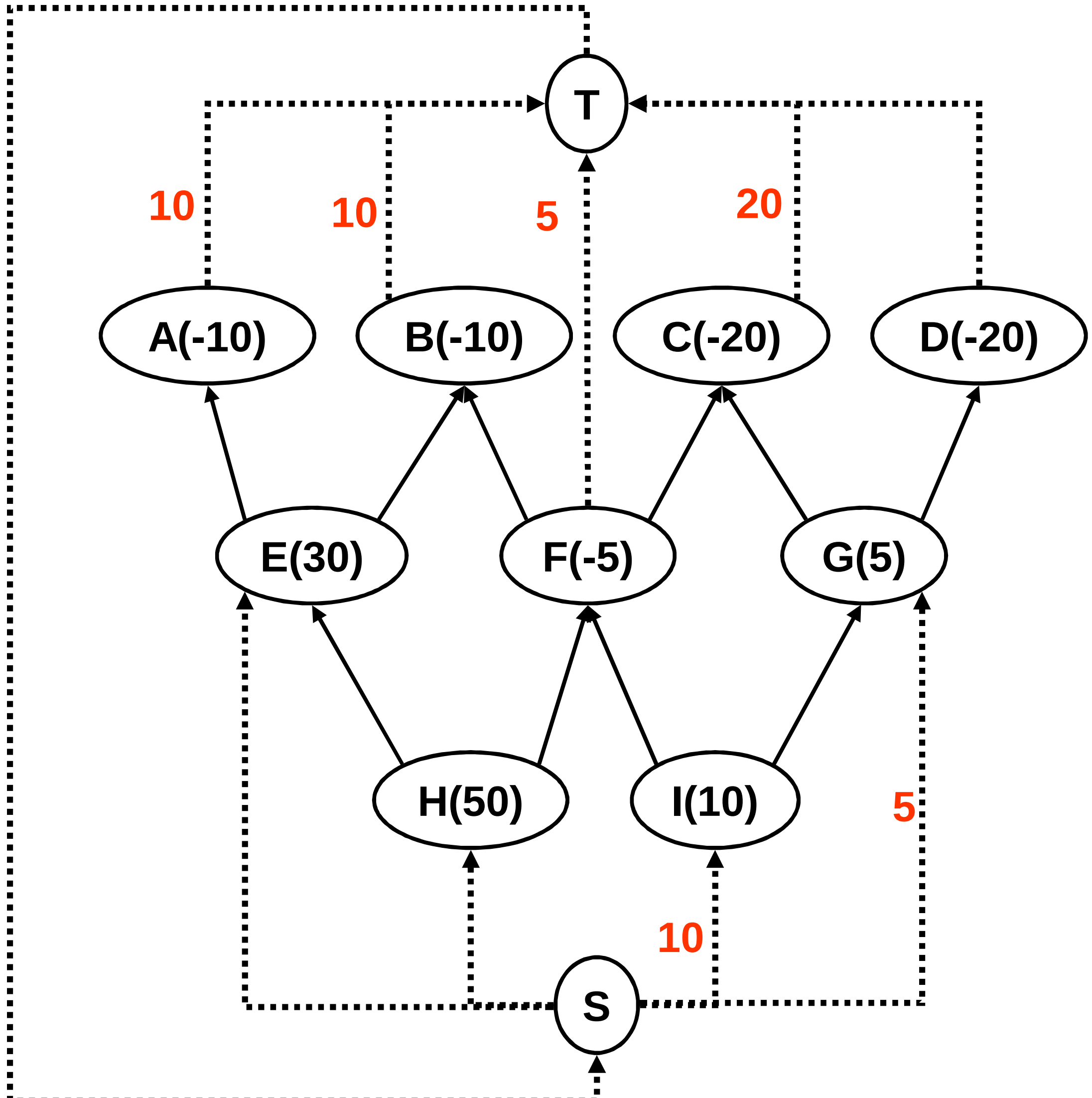
Flows shown on diagram

But what nodes are in the closure? And what's the weight?

Aside: the Max-Flow Min-Cut Theorem

- In a maximum flow problem, the optimal flow is equal to the capacity of the minimum "cut"
  - A *cut* is a set of arcs that divides the network into two sets of nodes, one containing the source (S) and the other the sink (T)
  - Call these sets of nodes N_1 and N_2
  - Each arc in the cut set has one endpoint in N_1 and another in N_2
- Consequences:
  - Solving the max flow problem also gives the minimal set of arcs that can "disconnect" the network
  - The arcs in the cut will all be at their upper bounds
  - A large network can have many cutsets
  - May have to resort to a separate algorithm to find them all

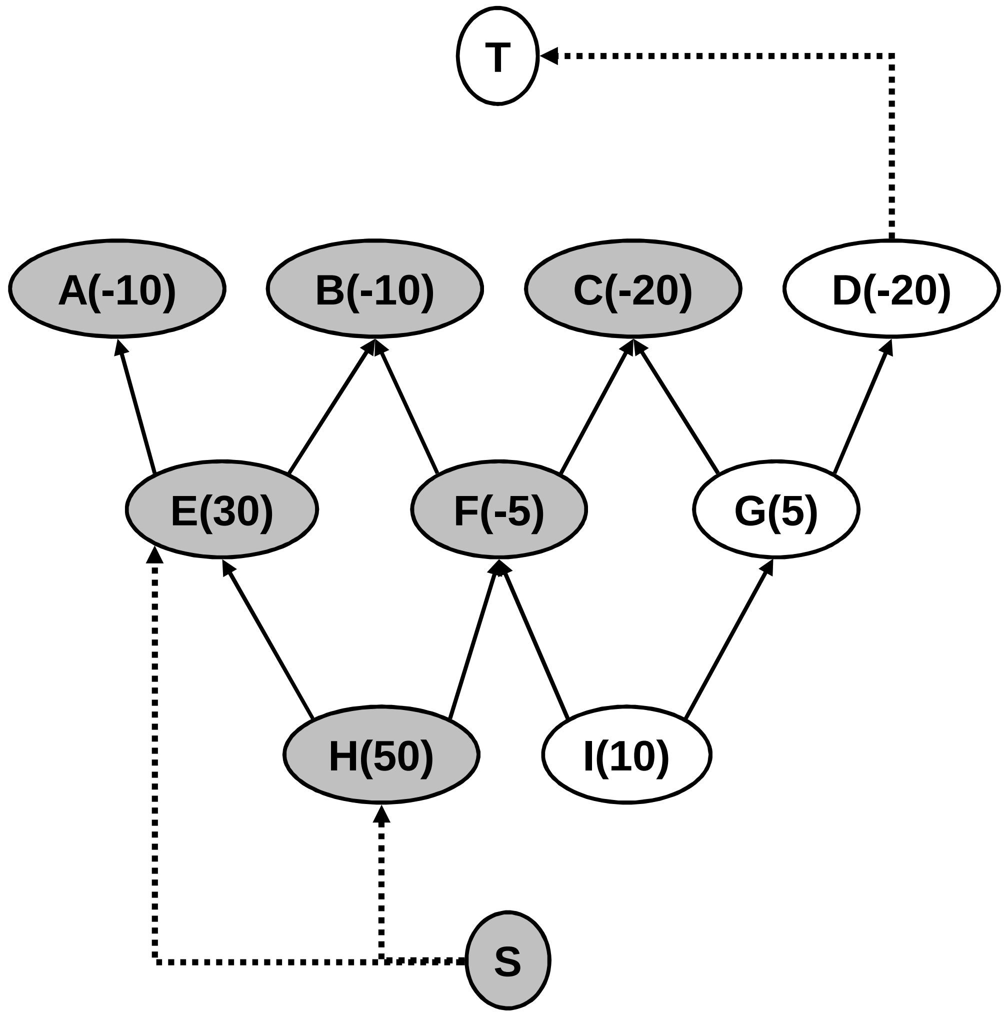
Finding the Min Cut in the Closure Problem



Marked arcs are at their upper bounds

Note that the sum of those bounds *is* the max flow

Delete the Arcs in the Min Cut



Now we see the partition of the nodes ...

Which partition is the maximum weight closure? And what is its weight?

Some Final Notes

- We can solve the max weight closure problem directly:

$$\max\ z = \sum_i W_i x_i$$
$$\text{subject to}$$
$$x_i \le x_j \text{ for all } i, j \in ARCS(i, j)$$
$$x_i \in \{0,1\} \text{ for all } i$$

- People convert it to a network because:
  - There are special max flow algorithms available that do not require expensive LP solvers
  - It's relatively easy to code these algorithms and they run quickly
- However, you must do added work to find the solution
- See http://128.32.125.151/riot/index.html (the Remote Interactive Optimization Testbed) website for a demo

The Critical Path Method (CPM)

- Recent evolution of project scheduling
  - Methodology depended on who was in charge
  - After WW I, the Gantt (bar) chart became a popular method
  - But, bar charts had limited ability to depict complex relationships
- DuPont and Remington Rand Univac developed a new method in the late 1950's
  - Approach was to depict the project as a network
  - Aim was provide a means to investigate tradeoffs in project cost and duration
  - Came to be known as CPM

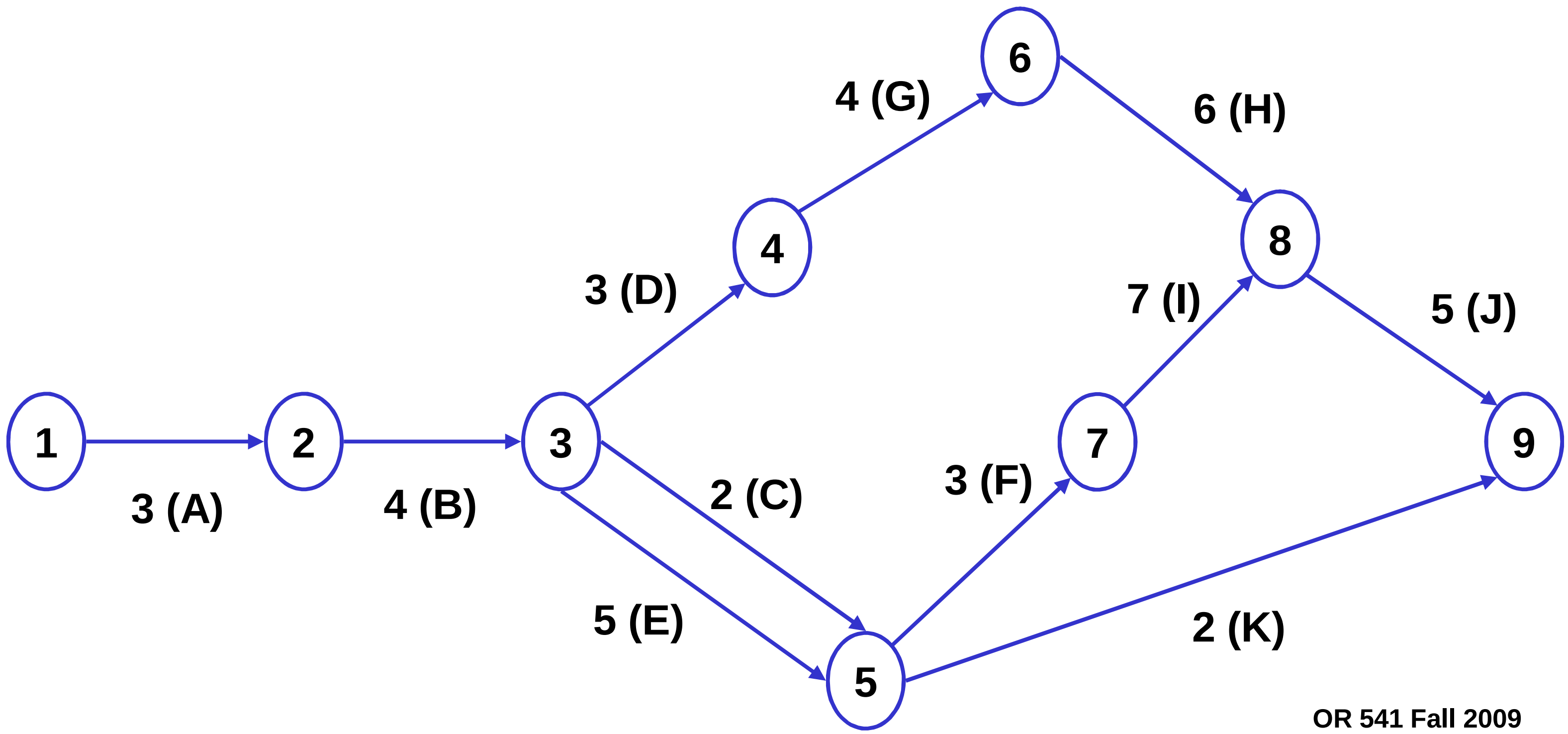
CPM Formulation

- CPM is essentially a longest-path problem (and can be depicted as an MCNFP)
- Consider the following example (from Schrage):

Activity	Job #	Time	Predecessors
Dig basement	A	3	none
Pour foundation	B	4	A
Pour basement floor	C	2	B
Install floor joists	D	3	B
Install Walls	E	5	B
Install rafters	F	3	C,E
Install flooring	G	4	D
Rough interior	H	6	G
Install roof	I	7	F
Finish interior	J	5	I,H
Landscape	K	2	C,E

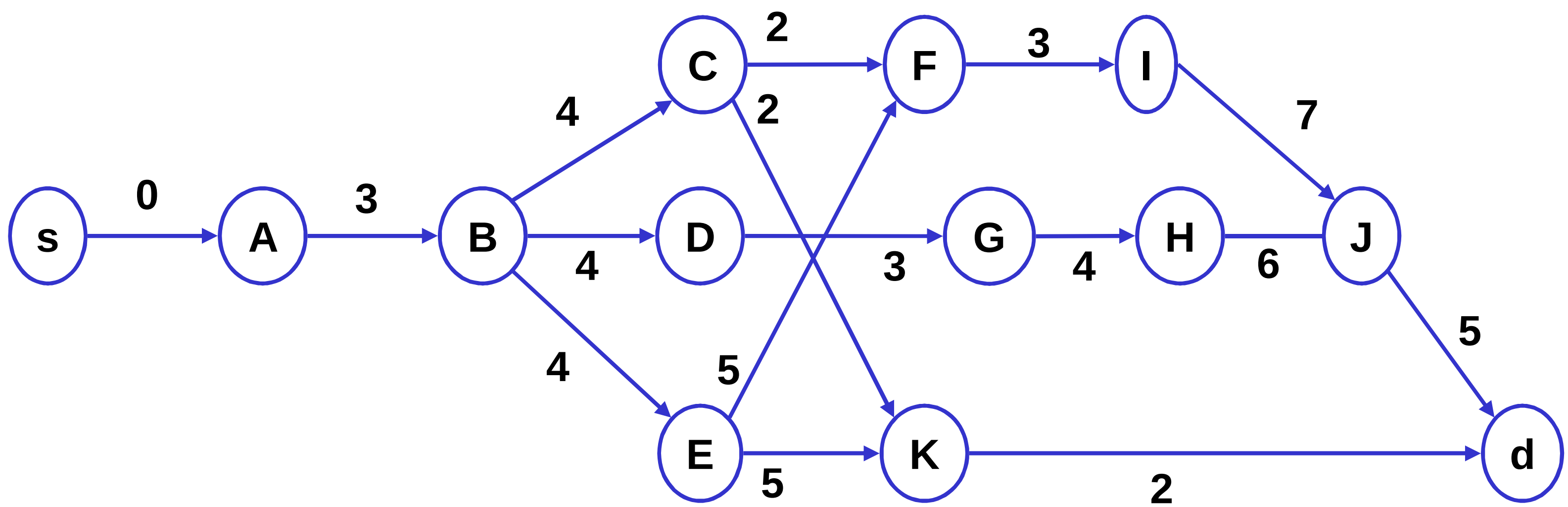
Network Representation (Activity-on-Arc, or AOA)

- Nodes represent precedences
- Arcs represent activities and completion times
- Project time is the *longest* path from 1 to 9
- What's the critical path?



A Better Representation: Activity on Node (AON)

- Here is the same problem with the nodes as activities



- This representation is *far superior* to AOA
  - AOA frequently requires dummy arcs to depict precedences
  - Minimizing the number of dummy arcs is a difficult problem
  - We will *not* use AOA representations in this course

CPM as a Longest Path Problem

- Just maximize the shortest path formulation:

$$\max\ z = \sum_{i,j \in ARCS(i,j)} C_{ij} * x_{ij}$$

subject to

$$\left[\sum_{j \in ARCS(i,j)} x_{ij}\right] - \left[\sum_{j \in ARCS(j,i)} x_{ji}\right] = \begin{cases} 1, i = s \\ 0, i \neq s \text{ and } i \neq d \\ -1, i = d \end{cases} \begin{matrix} (u_s) \\ (u_i) \\ (u_d) \end{matrix}$$

$$0 \leq x_{ij} \leq 1 \text{ for all } ARCS(i, j)$$

- However, we will work (for now) with the *dual* of this problem
  - The indicies ***i, j*** (with start ***s*** and finish ***d***) now represent jobs
  - The variable u_i is the start time for each job
  - Let C_i be the completion time of job ***i***

Dual Formulation

- Here's what the dual looks like:

$$
\begin{aligned}
&\min z = u_d - u_s \\
&\text{subject to} \\
&u_j - u_i \geq C_i \text{ for all } i, j \in ARCS(i, j) \\
&u_i \text{ unrestricted for all } i
\end{aligned}
$$

- The dual is not a network!
  - The total time is the difference between $\boldsymbol{u_d}$ and $\boldsymbol{u_s}$
  - The rest of the constraints enforce precedences, completion times
  - The dual is easier to formulate (and extend) than the MCNFP
- This formulation *does* let us get at what the original researchers wanted to investigate, though ...

Project Crashing

- Addresses trades between expenditures, completion time
- Assume that:
  - You know the cost per unit time to "crash" a job (***CC_i***)
  - You know the minimum job completion time (***MIN_i***)
  - ***TOT*** is the total desired project time
- Formulation, where ***cr_i*** is the amount a job is crashed:

$$
\begin{aligned}
&\min\ z = \sum_i CC_i * cr_i \\
&\text{subject to} \\
&u_d - u_s \le TOT \\
&u_j \ge u_i + C_i - cr_i \text{ for all } i, j \in ARCS(i, j) \\
&u_i \text{ unrestricted for all } i \\
&0 \le cr_i \le C_i - MIN_i
\end{aligned}
$$

Just-In-Time Scheduling

- In this model, some jobs must start within a certain amount of time of other jobs
- Let $\boldsymbol{S_{ij}}$ be the max length of time between the start of job ***i*** and the start of job ***j***
- How do we modify the formulation to handle this?

$$\min z = u_d - u_s$$

subject to

$$u_j - u_i \geq C_i \text{ for all } i, j \in ARCS(i, j)$$

$$u_j \leq u_i + S_{ij} \text{ for all } i, j \text{ with } S_{ij} \geq 0$$

$$u_i \text{ unrestricted for all } i$$

Another Twist

- Suppose instead we penalize the time difference between the completion of job ***i*** and the start of job ***j***
- Let:
  - P_{ij} be the late penalty per unit time
  - ***TOT*** be the total desired project time
- The following formulation minimizes these penalties:

$$
\begin{aligned}
&\min\ z = \sum_{i,j \in ARCS(i,j)} \left(u_j - u_i - C_i\right) * P_{ij} \\
&\text{subject to} \\
&u_d - u_s \le TOT \\
&u_j - u_i \ge C_i \text{ for all } i,\ j \in\ ARCS(i,\ j) \\
&u_i \text{ unrestricted for all } i
\end{aligned}
$$

How Do You Find the Critical Path?

- Suppose you solve the example problem in MPL
- You get task start times, but don't know which ones are critical
- The key is to look at the dual values of the constraints, which represent the arcs
- Any arc with a *nonzero dual value* is on the critical path

i	j	Slack	Shadow Price
s	A	0	1
A	B	0	1
B	C	-3	0
B	D	-2	0
B	E	0	1
C	F	0	0
C	K	-13	0
D	G	0	0
E	F	0	1
E	K	-13	0
F	I	0	1
G	H	0	0
H	J	0	0
I	J	0	1
J	d	0	1
K	d	0	0

MPL Code

```
TITLE
   CPM; {  Schrage CPM example; MPL must be }
           {   in case sensitive mode! }

INDEX
   node  := (s,A,B,C,D,E,F,G,H,I,J,K,d);
   i        := node;
   j        := node;

DATA

   { prec is used to define precedence arcs }

   prec[i,j] :=

   [s,A,1,
    A,B,1,
    B,C,1, B,D,1, B,E,1,
    C,F,1, C,K,1,
    D,G,1,
    E,F,1, E,K,1,
    F,I,1
    G,H,1,
    H,J,1,
    I,J,1,
    J,d,1,
    K,d,1];

 { note 0 (dummy) duration times for s and d }

    dur[i] := (0,3,4,2,3,5,3,4,6,7,5,2,0);


VARIABLES

   u[node];

MODEL

   min span = u["d"] - u["s"];

SUBJECT TO

   precedence[i,j] where prec[i,j]>0:

     u[node:=j] - u[node:=i] > dur[i];

END
```