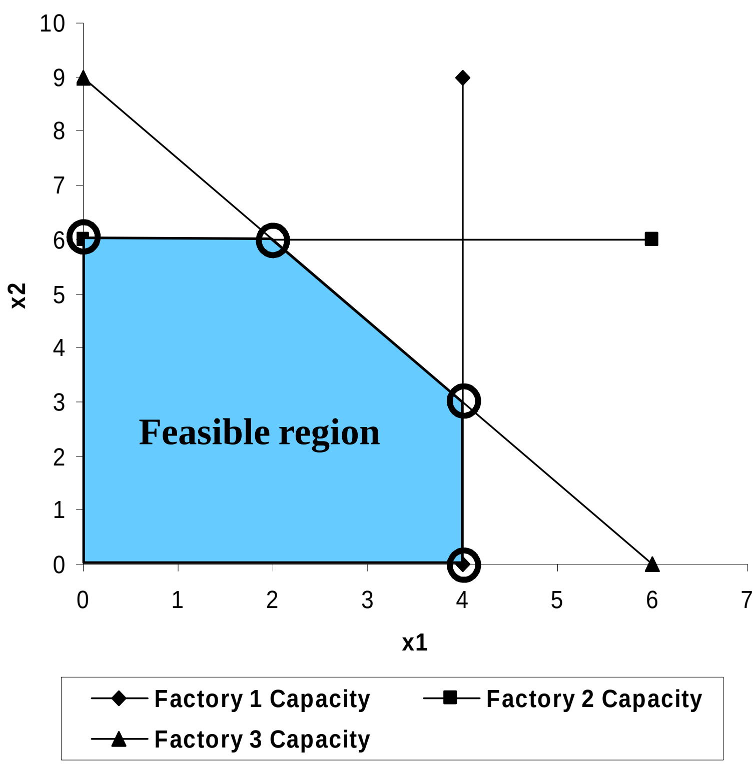
Simplex Algorithm: Tableau and Pivoting

- For this example, I'll show what's going on algebraically and graphically
- Stick with the Wyndor Glass problem:



$$\max \quad Z = 3x_1 + 5x_2$$

subject to:

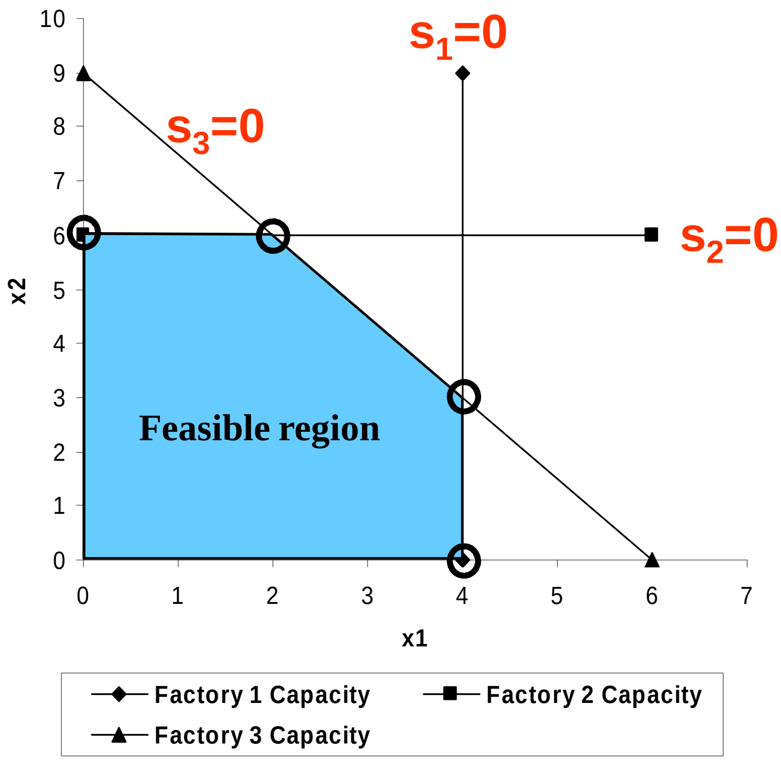
$$x_1 \le 4 \text{ (plant 1 capacity)}$$

$$2x_2 \le 12 \text{ (plant 2 capacity)}$$

$$3x_1 + 2x_2 \le 18 \text{ (plant 3 capacity)}$$

$$x_1, x_2 \ge 0 \text{ (no negative production)}$$

Step 1: Convert Problem to Standard Form



$$\max \quad Z = 3x_1 + 5x_2, \text{ or}$$

$$Z - 3x_1 - 5x_2 = 0$$

subject to :

$$\begin{aligned} x_1 \quad\quad\quad + s_1 \quad\quad\quad &= 4 \\ 2x_2 \quad\quad + s_2 \quad &= 12 \\ 3x_1 + 2x_2 \quad\quad\quad + s_3 &= 18 \end{aligned}$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

Step 2: Arrange Into Simplex Tableau

Equation Form

z	$-3x_1$	$-5x_2$				=0
	x_1		$+s_1$			=4
		$2x_2$		$+s_2$		=12
	$3x_1$	$+2x_2$			$+s_3$	=18

Tableau Form

Row	z	x1	x2	s1	s2	s3	RHS	BV
0	1	-3	-5	0	0	0	0	z
1		1	0	1	0	0	4	s1
2		0	2	0	1	0	12	s2
3		3	2	0	0	1	18	s3

Decoding the Tableau

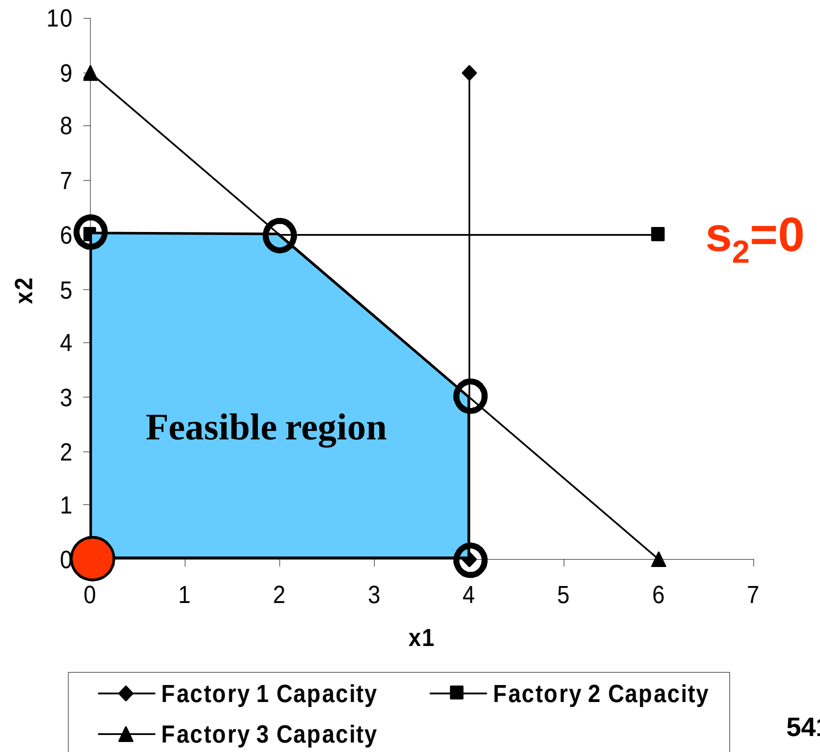
Row	z	x1	x2	s1	s2	s3	RHS	BV
0	1	-3	-5	0	0	0	0	z
1		1	0	1	0	0	4	s1
2			2	0	1	0	12	s2
3		3	2	0	0	1	18	s3

Basic Solution

Values of Basic Variables

Objective Function Value

Nonbasic Variables

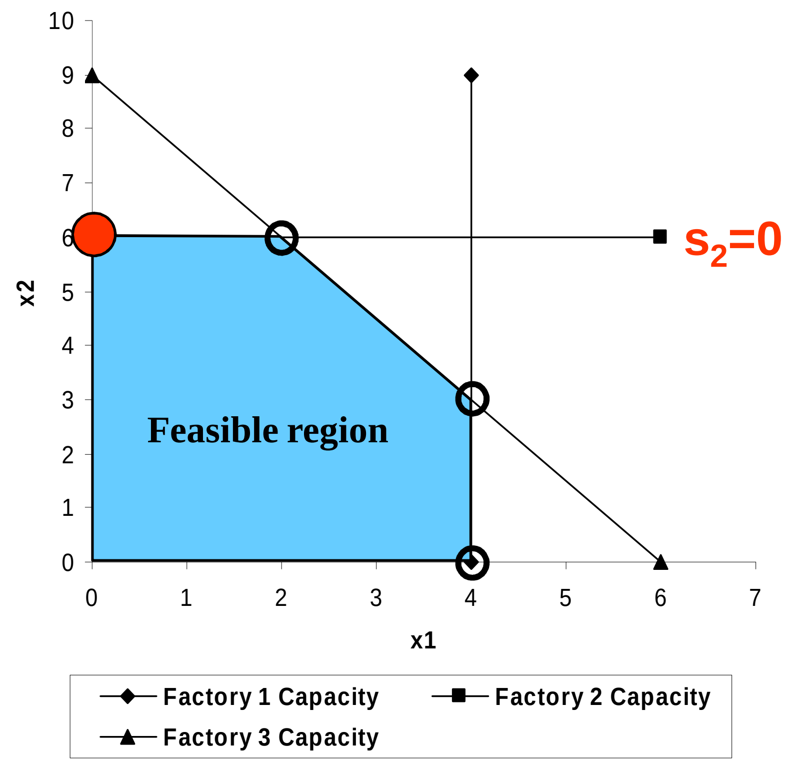


Deciding How To Move

- We want to go to an *adjacent* extreme point
- We have to determine:
  - Which variable leaves the basis (solution)
  - Which variable enters the basis (solution)
  - The value of the entering variable
  - The resulting value of the objective function
- How to think about this
  - Variables in the basis are *dependent*; think of their values as *data*
  - Nonbasic variables are *independent*
  - Right now, the objective function, in terms of the independent variables only, is $z = 3x_1 + 5x_2$

Choosing the Entering Variable

- We'd like as much improvement as possible
- From calculus:
  - $z = 3x_1 + 5x_2, \frac{dz}{dx_1} = 3, \frac{dz}{dx_2} = 5$
- So, which is the better choice?
  - Note also the economic interpretation; for every unit increase of x_2, we get $5 additional profit
- How far can we go? Which variable will exit?



Simplex "Pivoting"

- Swapping a variable into the basis is called a *pivot*
- This operation has to maintain feasibility, for both the leaving and entering variables
- Method:
  - Take rows out of the tableau containing x_2
  - Write x_2 in terms of the current basic variables
  - Determine the maximum x_2 can increase

$$\begin{aligned} 2x_2 \quad + s_2 \quad &= 12, \Rightarrow x_2 = 6 - \frac{s_2}{2} \\ 2x_2 \quad\quad + s_3 &= 18, \Rightarrow x_2 = 9 - \frac{s_3}{2} \end{aligned}$$

- **What's the most x_2 can increase?**
- **What if x_2's coefficient was negative?**

Doing the Pivot Via the Tableau

- So, x_2 comes in, and s_2 leaves; the cell is the pivot
- Doing this operation in the tableau is the so-called "minimum ratio test"

Row	z	x1	x2	s1	s2	s3	RHS	BV	Ratio
0	1	-3	-5	0	0	0	0	z	
1		1	0	1	0	0	4	s1	
2		0	2	0	1	0	12	s2	6
3		3	2	0	0	1	18	s3	9

- What's the objective function value now?
- How do we update the tableau?

Tableau Updating

- Do elementary row operations in the tableau to make x_2's column look like s_2's
  - Divide pivot row by pivot element
  - Add/subtract multiples of pivot row to get 0's above and below the pivot element

Row 2 = Row 2/2

Row	z	x1	x2	s1	s2	s3	RHS	BV
0	1	-3	-5	0	0	0	0	z
1		1	0	1	0	0	4	s1
2		0	1	0	0.5	0	6	x2
3		3	2	0	0	1	18	s3

Remaining Row Operations

Row 3 = Row 3 - 2*Row 2

Row	z	x1	x2	s1	s2	s3	RHS	BV
0	1	-3	-5	0	0	0	0	z
1		1	0	1	0	0	4	s1
2		0	1	0	0.5	0	6	x2
3		3	0	0	-1	1	6	s3

Row 0 = Row 0 +5*Row 2

Row	z	x1	x2	s1	s2	s3	RHS	BV
0	1	-3	0	0	2.5	0	30	z
1		1	0	1	0	0	4	s1
2		0	1	0	0.5	0	6	x2
3		3	0	0	-1	1	6	s3

What We Did (Linear Algebra)

- The initial BFS was s_1, s_2, s_3, and the system was:
  - $$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}s_1\\s_2\\s_3\end{bmatrix}=\begin{bmatrix}4\\12\\18\end{bmatrix}\Rightarrow\begin{bmatrix}s_1\\s_2\\s_3\end{bmatrix}=\begin{bmatrix}4\\12\\18\end{bmatrix}$$
- Now, the BFS is s_1, x_2, s_3, and the system is:

$$\begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 2 & 1\end{bmatrix}\begin{bmatrix}s_1\\x_2\\s_3\end{bmatrix}=\begin{bmatrix}4\\12\\18\end{bmatrix}\Rightarrow\begin{bmatrix}s_1\\x_2\\s_3\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\0 & .5 & 0\\0 & -1 & 1\end{bmatrix}\begin{bmatrix}4\\12\\18\end{bmatrix}=\begin{bmatrix}4\\6\\6\end{bmatrix}$$

- **All the tableau does is provide bookkeeping for a sequence of solutions of the form Ax = b**
  - **Does the inverse look familiar?**

The Next Pivot

- Note that the equation for z changes, because the *independent variables have changed*
  - $$z = 3x_1 - 2.5s_2 + 30$$
    $$\frac{dz}{dx_1} = 3, \frac{dz}{ds_2} = -2.5$$
- Which variable should enter?
- Which variable should exit?

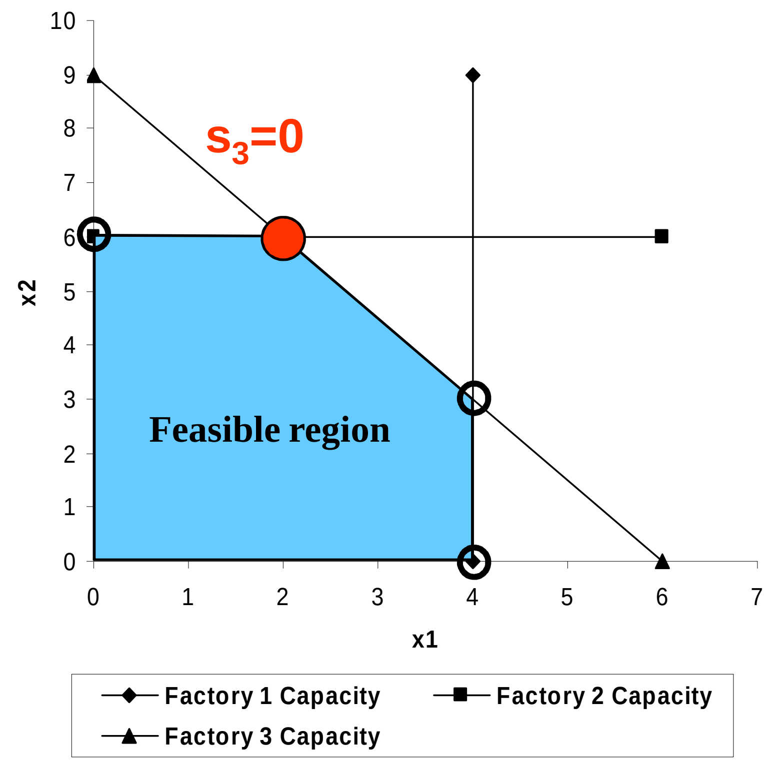


Tableau Operations

Min ratio test: s_3 goes out, as expected

Row	z	x1	x2	s1	s2	s3	RHS	BV	Ratio
0	1	-3	0	0	2.5	0	30	z	
1		1	0	1	0	0	4	s1	4
2		0	1	0	0.5	0	6	x2	
3		3	0	0	-1	1	6	s3	2

Row	z	x1	x2	s1	s2	s3	RHS	BV
0	1	-3	0	0	2.5	0	30	z
1		1	0	1	0	0	4	s1
2		0	1	0	0.5	0	6	x2
3		1	0	0	-0.333	0.333	2	x1

Row 3 = Row 3/3

Row	z	x1	x2	s1	s2	s3	RHS	BV
0	1	-3	0	0	2.5	0	30	z
1		0	0	1	0.333	-0.333	2	s1
2		0	1	0	0.5	0	6	x2
3		1	0	0	-0.333	0.333	2	x1

Row 1 = Row 1 - Row 3

The Finale

Row	z	x1	x2	s1	s2	s3	RHS	BV
0	1	0	0	0	1.5	1	36	z
1		0	0	1	0.333	-0.333	2	s1
2		0	1	0	0.5	0	6	x2
3		1	0	0	-0.333	0.333	2	x1

Row 0 = Row 0 + 3*Row 3

$$z = -1.5s_2 - s_3 + 36$$

$$\frac{dz}{ds_2} = -1.5, \frac{dz}{ds_3} = -1$$

NO IMPROVEMENT POSSIBLE; SOLUTION IS OPTIMAL

Some Parting Questions

- The coefficients in row 0 for the variables are commonly called “reduced costs.” Why?
- What’s the stopping rule for simplex?
- Take a look at the last 3 columns in the final tableau. If we multiply that matrix by the original RHS, what do you think we get?
- The last 3 columns in the final tableau are the inverse of some matrix. What is that matrix?
- If we could add 1 more unit of resource to one of the constraints, which one would we add it to? Can you tell from the tableau?

Simplex II: Other Stopping Conditions

- Remember that an algorithm needs 3 elements:
  - A way to start
  - A way to iterate
  - A way to stop
- We have covered the way simplex iterates, and the normal stopping condition
- There are three other stopping conditions to consider
  - Multiple optimal solutions
  - Unbounded solution
  - Degenerate optimal solution
- Why don't we have an "infeasible problem" case?

A Benign Case: Alternative Optima

- Suppose we change the Wyndor Glass problem as follows:

$$\max \quad Z = 3x_1 + 2x_2, \text{ or}$$

$$Z - 3x_1 - 2x_2 = 0$$

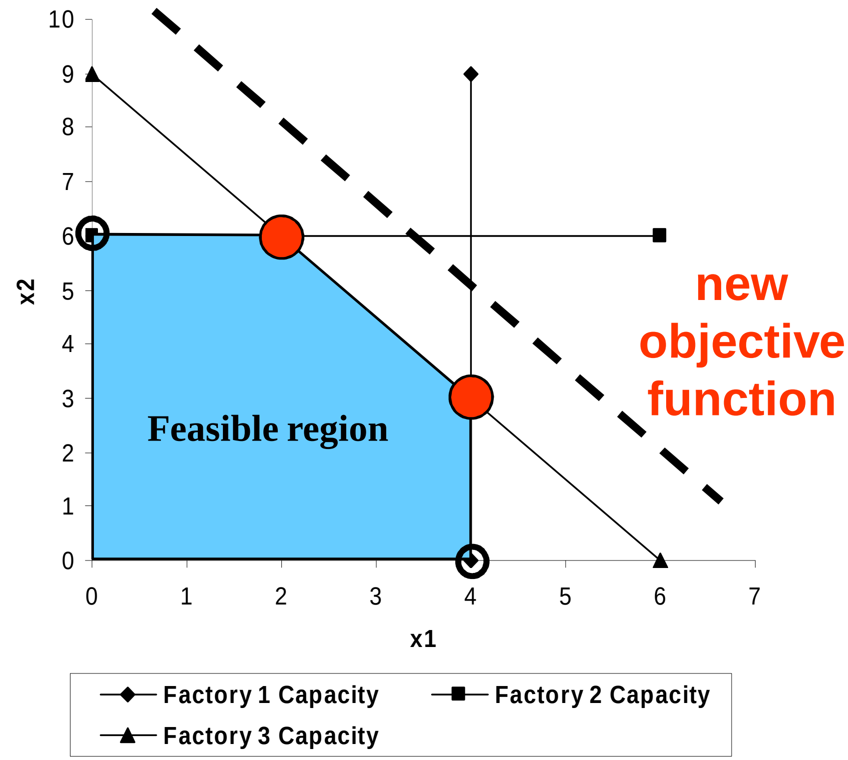
subject to :

$$x_1 + s_1 = 4$$

$$2x_2 + s_2 = 12$$

$$3x_1 + 2x_2 + s_3 = 18$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0$$



- What did we do?

The Final Tableau

Row	z	x1	x2	s1	s2	s3	RHS	BV
0	1	0	0	0	0	1	18	z
1		0	0	1	0.333	-0.333	2	s1
2		0	1	0	0.5	0	6	x2
3		1	0	0	-0.333	0.333	2	x1

- The reduced cost of s_2 is 0, but it's nonbasic
  - Bringing it in wouldn't hurt the solution
  - But it wouldn't help, either
  - What happens if we do pivot it in?
  - Can we pivot it in? What variable exits?

After the Pivot

- This is a case of *alternative optima*
  - The points **(2,6)** and **(4,3)** give the same value for **z**
  - **Any point on the line segment is optimal**
  - Swapping $\mathbf{s_1}$ and $\mathbf{s_2}$ moves from one extreme point to another

Row	z	x1	x2	s1	s2	s3	RHS	BV
0	1	0	0	0	0	1	18	z
1		0	0	3	1	-1	6	s2
2		0	1	-1.5	0	0.5	3	x2
3		1	0	1	0	0	4	x1

Practical Advice: Alternative Optima

- You should look at your solution for these situations
- Not easy (particularly in large-scale problems) to compute all alternative optimal extreme points
  - Variable approach: convert z to a constraint, then maximize the variable (or sum of variables) not in the solution
  - Constraint approach: force constraints to equality, see what happens
- Normally a signal to do more work refining the problem
  - There are normally other conditions to differentiate solutions
  - In Wyndor Glass, **(4,3)** might be better because it's a "more balanced" production scheme

Unbounded Solutions

- Suppose we run into a tableau like the one below:

Row	z	x1	x2	s1	s2	s3	RHS	BV
0	1	0	-3	2	0	0	32	z
1		0	0	3	1	0	7	s2
2		0	-4	-1.5	0	1	3	s3
3		1	-5	1	0	0	8	x1

- Where do we pivot? The equations are:

$$-4x_2 \qquad + s_3 \qquad = 3, \Rightarrow x_2 = \frac{3 + s_3}{4}$$

$$-5x_2 \qquad\qquad + x_1 = 8, \Rightarrow x_2 = \frac{8 + x_1}{5}$$

- Nothing wants to be driven to 0!

What Negative Column Coefficients Mean

- If there is negative coefficient in a nonbasic variable column, the feasible region is unbounded
- If there is NO CHOICE of leaving variable (all pivot elements nonpositive) the problem is unbounded
- What happened? You either:
  - Omitted a variable in a constraint
  - Inadvertently added a variable in the objective
  - Entered a coefficient wrong
- Chasing this down can drive you crazy in a large problem - SO BOUND ALL YOUR VARIABLES
  - Aside: putting bounds on variables makes simplex much more efficient

Degenerate Optimal Solutions

- An objective function coefficient of 0 does not always mean there are multiple optima
- Consider a modification of Wyndor Glass:

$$\max \quad Z = 3x_1 + 5x_2, \text{ or}$$

$$Z - 3x_1 - 5x_2 = 0$$

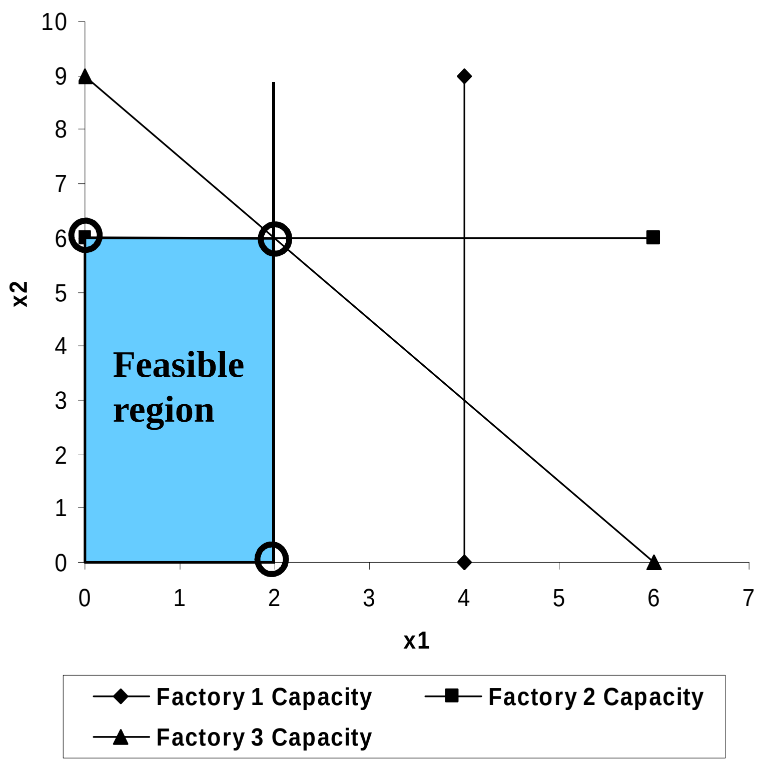
subject to :

$$x_1 + s_1 = 2$$

$$2x_2 + s_2 = 12$$

$$3x_1 + 2x_2 + s_3 = 18$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0$$



Finale Tableaus

- There are multiple representations of the same point!

Row	z	x1	x2	s1	s2	s3	RHS	BV
0	1	0	0	0	0	1	36	z
1		0	0	1	0.333	-0.333	0	s1
2		0	1	0	0.5	0	6	x2
3		1	0	0	-0.333	0.333	2	x1

Row	z	x1	x2	s1	s2	s3	RHS	BV
0	1	0	0	0	0	1	36	z
1		0	0	3	1	-1	0	s2
2		0	1	-1.5	0	0.5	6	x2
3		1	0	1	0	0	2	x1

- Such as case is called a degenerate basic solution, and it’s very common

Cycling

- If it's common to get multiple representations of the same point, can simplex get stuck?
  - Is it possible for the method to just swap among a number of solutions?
- Answer - YES: called cycling
  - Winston says (p. 162) "in practice, however, cycling is an extremely rare occurrence"
  - HE IS WRONG, particularly for network problems and huge LPs
- Some history
  - In the early days of LP, cycling was rare due to small problem sizes and computer round-off error
  - Precision arithmetic and large-scale problems reintroduced it

Cycling and Stalling

- Cycling - simplex gets stuck among a set of BFS's, with no solution improvement
- Stalling - a long sequence of degenerate pivots, with no solution improvement
- So: a cycle is a stall that doesn't quit (or else you get upset and shut down the solve, which has the same effect)
- Every commercial solver devotes a great deal of code to anti-stalling (and cycling) techniques
- If it's rare (as Winston claims), why is everyone worried about it?

Bland's Rule (1977) for Cycling Prevention

- Here's a very simple technique:
  1. Give each of the **n** variables an index number
  2. At each iteration, look at all nonbasic variables with a favorable reduced cost
  3. Enter the one with the smallest index
  4. If there is a tie in the ratio test for the leaving variable, choose the one with the smallest index
- Why does this break a cycle?
  - In a cycle, some variable $\mathbf{x_j}$ must enter and leave the basis
  - However, if it leaves, it must be replaced by some variable with an index *higher* than **j** that was nonbasic when $\mathbf{x_j}$ entered
  - Essentially forces simplex to avoid previously-examined variables

Some Methods Used by Commercial Solvers

- Stalling prevention
  - Problem scaling
  - Alternative reduced cost schemes (e.g. DEVEX, steepest edge)
- Stalling cures
  - Most solvers monitor progress of objective, and turn on procedures if they detect stalling
  - Bland’s Rule and derivatives
  - Perturbation (artificially moving variables/constraints off bounds)
  - You will see evidence in solver reports of this occurring
- EVERY NONTRIVIAL PROBLEM WILL HAVE SOME SET OF DEGENERATE PIVOTS

Aside: Unrestricted Variables

- Winston (Sec. 4.14) conflates several ideas
  - Solvers do NOT handle unrestricted variables the way he suggests
  - He does, however, raise an important issue
- Yet another Wyndor Glass modification
  - Suppose we are penalized $2/unit for every unit of factory capacity that is either over or under the target of 18
  - We cleverly decide to use an unrestricted variable, y_1:

$$
\begin{aligned}
&\max \quad Z = 3x_1 + 5x_2 - 2y_1 \\
&\text{subject to:} \\
&x_1 + s_1 = 4 \\
&2x_2 + s_2 = 12 \\
&3x_1 + 2x_2 + y_1 = 18 \\
&x_1, x_2, s_1, s_2 \ge 0 \\
&y_1 \text{ unrestricted}
\end{aligned}
$$

We Load the Problem, and ...

- The solution is:
  - **$x_1 = 6, x_2 = 4, y_1 = -6, z = 54$**
  - Is this right? Should be **z** = 3*4+5*6-2*6 = **30**!
- We must have messed up the objective
  - Change it to $3x_1 + 5x_2 + 2y_1$
  - New answer is **$x_1 = 0, x_2 = 6, y_1 = 6, z = 42$**
  - Is this right? Should be **z** = 3*0+5*6-2*6 = **18**!
- **What the #$%@^&!! is going on here?**

Here's the Issue

- If a solver sees an unrestricted variable, it will (generally) substitute it out of the problem

Set $y_1 = 18 - 3x_1 - 2x_2$, eliminate 3rd constraint

$$
\begin{aligned}
&\max \quad Z = 3x_1 + 5x_2 - 2(18 - 3x_1 - 2x_2) \\
&= 9x_1 + 9x_2 - 36 \\
&\text{subject to:} \\
&x_1 \qquad + s_1 \qquad = 4 \\
&\qquad 2x_2 \qquad + s_2 \qquad = 12 \\
&x_1, x_2, s_1, s_2 \geq 0
\end{aligned}
$$

- Now we see what it did! It turns out our formulation is WRONG

The Fix (and What Winston Does)

- It turns out that the objective function really is:

$$\max \quad Z = 3x_1 + 5x_2 - 2|y_1|$$

- We need to define two new variables to do this:
  - y_1: number of units below 18
  - y_2: number of units above 18

$$\begin{aligned}
&\max \quad Z = 3x_1 + 5x_2 - 2y_1 - 2y_2 \\
&\text{subject to:} \\
&x_1 + s_1 = 4 \\
&2x_2 + s_2 = 12 \\
&3x_1 + 2x_2 + y_1 - y_2 = 18 \\
&x_1, x_2, s_1, s_2, y_1, y_2 \geq 0
\end{aligned}$$

- We can guarantee that y_1 and y_2 will never both be in the solution! Why?

Simplex III: Finding an Initial BFS

- We know how to iterate, and how to stop
- But, how do we start?
- Consider the following LP, not in standard form:

$$
\begin{aligned}
&\min \quad z = x_1 + 4x_2 - x_4 \\
&\text{subject to :} \\
&-x_1 + 2\mathrm{x}_2 - x_3 + \mathrm{x}_4 \le 2 \\
&\ 2x_1 + x_2 + 2x_3 \text{-} 2x_4 = 4 \\
&x_1 - 3x_3 + x_4 \ge 2 \\
&x_1, x_2, x_4 \ge 0,\ x_3 \text{ unrestricted}
\end{aligned}
$$

Convert to Standard (Max) Form, But Then?

- No obvious starting solution
  - There's no identity matrix for a basis
  - If we try to put in s_2 directly, its value is negative (a violation)

$$\begin{aligned}
&\max \quad z = -x_1 - 4x_2 + x_4 \\
&\text{subject to :} \\
&-x_1 + 2x_2 - x_3 + x_4 + s_1 = 2 \\
&2x_1 + x_2 + 2x_3 - 2x_4 = 4 \\
&x_1 - 3x_3 + x_4 - s_2 = 2 \\
&x_1, x_2, x_4, s_1, s_2 \geq 0, x_3 \text{ unrestricted}
\end{aligned}$$

- Cure: add "artificial variables"
  - Only need artificials for the last two constraints
  - Yields starting solution of **s_1=2, a_1=4, a_2=2**

$$\begin{aligned}
&\max \quad z = -x_1 - 4x_2 + x_4 \\
&\text{subject to :} \\
&-x_1 + 2x_2 - x_3 + x_4 + s_1 = 2 \\
&2x_1 + x_2 + 2x_3 - 2x_4 + a_1 = 4 \\
&x_1 - 3x_3 + x_4 - s_2 + a_2 = 2 \\
&x_1, x_2, x_4, s_1, s_2, a_1, a_2 \geq 0 \\
&x_3 \text{ unrestricted}
\end{aligned}$$

How Do We Get Rid of These Artificials?

- Textbook approaches:
  - Give them a big penalty in the objective function, hope they go away (Big-M method)
  - Minimize their sum, and throw them away when done (Two-Phase)
- Two-Phase applied to our example:

$$\min\ y = a_1 + a_2, \text{ or } \max -y = -a_1 - a_2$$

$$\text{subject to:}$$

$$-x_1 + 2\mathrm{x}_2 - x_3 + x_4 + s_1 \qquad = 2$$

$$2x_1 + x_2 + 2x_3 - 2x_4 \qquad + a_1 \qquad = 4$$

$$x_1 - 3x_3 + x_4 - s_2 \qquad + a_2 = 2$$

$$z = -x_1 - 4x_2 + x_4$$

$$x_1, x_2, x_4, s_1, s_2, a_1, a_2 \geq 0$$

$$x_3,\ z \text{ unrestricted}$$

Set Up a Phase I Tableau

- We will carry the original objective function row, plus add a row for the Phase I objective

Row	z	y	x1	x2	x3	x4	s1	s2	a1	a2	RHS
z	1		1	4	0	-1	0	0	0	0	0
Ph I		1	0	0	0	0	0	0	1	1	0
1		s1	-1	2	-1	1	1	0	0	0	2
2		a1	2	1	2	-2	0	0	1	0	4
3		a2	1	0	-3	1	0	-1	0	1	2

- Note we can't start yet; we have to "clear out" the Phase I objective row (just subtract Row 2 and Row 3)
- For the rest of the pivots, we will transform the z row as well

Pivot Sequence

Clear Ph 1 Row: Ph1 = Ph1 - Row 2 - Row 3

Row	z	y	x1	x2	x3	x4	s1	s2	a1	a2	RHS
z	1		1	4	0	-1	0	0	0	0	0
Ph I		1	-3	-1	1	1	0	1	0	0	-6
1		s1	-1	2	-1	1	1	0	0	0	2
2		a1	2	1	2	-2	0	0	1	0	4
3		a2	1	0	-3	1	0	-1	0	1	2

After a pivot in x1 column, row 3

Row	z	y	x1	x2	x3	x4	s1	s2	a1	a2	RHS
z	1		0	4	3	-2	0	1	0	-1	-2
Ph I		1	0	-1	-8	4	0	-2	0	3	0
1		s1	0	2	-4	2	1	-1	0	1	4
2		a1	0	1	8	-4	0	2	1	-2	0
3		x1	1	0	-3	1	0	-1	0	1	2

The Results of Phase I

Row	z	y	x1	x2	x3	x4	s1	s2	a1	a2	RHS
z	1		0	3.63	0	-0.5	0	0.25	-0.4	-0.3	-2
Ph I		1	0	0	0	0	0	0	1	1	0
1		s1	0	2.5	0	0	1	0	0.5	0	4
2		x3	0	0.13	1	-0.5	0	0.25	0.13	-0.3	0
3		x1	1	0.38	0	-0.5	0	-0.3	0.38	0.25	2

- OK - artificials out (**but what's wrong now?**)
- What does it mean if:
  - (Case A) At least one artificial stays in the solution at optimality?
  - (Case B) At least one artificial is in the basis, but is equal to 0, at optimality?

Outcomes

- Case A: if an artificial stays in, the problem is **infeasible**
- Case B: a couple things can happen
  - We can pivot all the degenerate artificials out of the basis; then we proceed as usual
  - Suppose some artificials remain at the 0 level
  - In the last case, it turns out that this is an indication that the rows (constraints) where the remaining artificials are basic are *redundant,* and can be thrown away

Final Thoughts on Getting an Initial Solution

- Why do you think implementing the Big-M method might be a problem? What is the advantage of Big-M?
- Many large LPs spend the majority of their time in Phase I, trying to get feasible
  - This happens when the model has lots of “chains” of relationships that must be satisfied
  - Models like this are prone to stalling in Phase I
- Most commercial solvers will let you provide an initial solution
  - If you have one (from a previous solve or via some heuristic), by all means use it!