Simple Bounds in the Dual

- Many problems have simple bounds on primal variables
  - How do these show up in the dual?
  - Also, what if we have simple bounds on the dual variables?
- Consider the following "elastic" LP:

$$\max\ z = c_1 x + c_2 s_1 - c_3 s_2$$

subject to

$$Ax + s_1 - s_2 = b$$

$$l \le x \le u$$

$$s_1, s_2 \ge 0$$

- In this LP, every constraint is really a "goal"
  - Objective function has rewards and penalties for deviations
  - The auxiliary variables are slacks ($\boldsymbol{s_1}$) and surpluses ($\boldsymbol{s_2}$)

The Dual of the Elastic LP

- The primal bounds end up in the dual objective, and the primal rewards/penalties become dual bounds

$$
\begin{aligned}
&\min\ y = w_1 b + w_2 u - w_3 l \\
&\text{subject to} \\
&\qquad w_1 A + w_2 - w_3 \geq c_1 \\
&\qquad c_2 \leq w_1 \leq c_3 \\
&\qquad w_2, w_3 \geq 0
\end{aligned}
$$

- This is a useful model when:
  - It is unclear what the RHS should be
  - It is unclear if the RHS can even be achieved **(FOOTSTOMP)**
  - You can estimate the feasible range of the shadow prices

Adding Constraints to an LP

- Suppose I have the following integer program:

$$\min\ z = 3x_1 + 4x_2$$

subject to

$$3x_1 + x_2 \geq 4,\ \text{or}\ 3x_1 + x_2 - s_1 = 4$$
$$x_1 + 2x_2 \geq 4,\ \text{or}\ x_1 + 2x_2 - s_2 = 4$$
$$x_1, x_2 \geq 0\ \text{and integer},\ s_1, s_2 \geq 0$$

- I employ the "prayer method" (solve as an LP and hope the answer's integral) and get:
- $x_1 = 4/5$, $x_2 = 8/5$
- Now what?

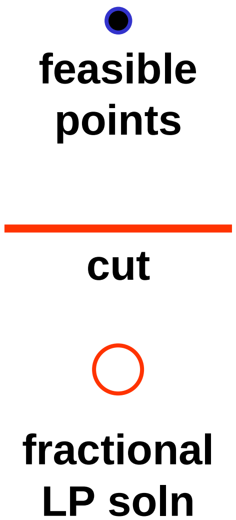
Adding a "Cut"

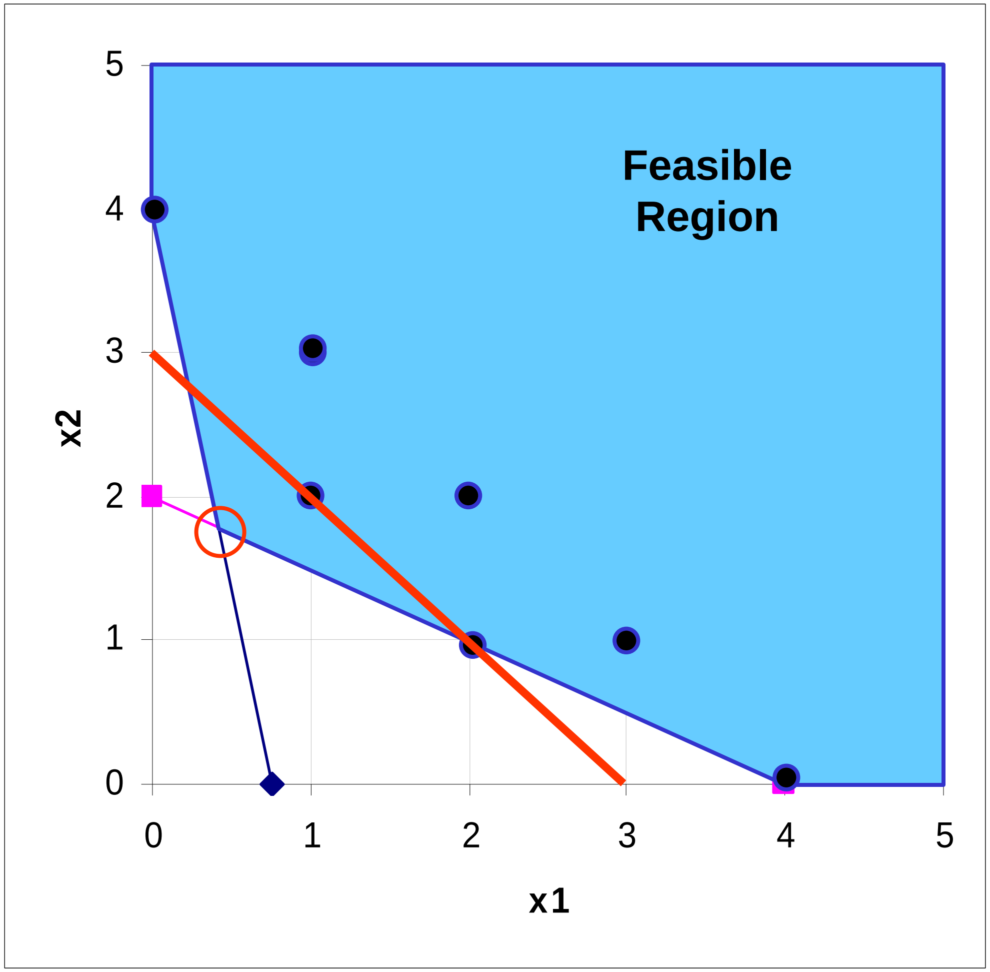
- I will now do something *very strange*; I add the following constraint to the model:

$$\frac{1}{5}s_1 + \frac{2}{5}s_2 \geq \frac{3}{5}, \quad \text{or} \quad \boxed{-\frac{1}{5}s_1 - \frac{2}{5}s_2 + s_3 = -\frac{3}{5}, \; s_3 \geq 0}$$

- This is called a "Gomory dual fractional cut"
  - What exactly is getting cut?
  - We will touch on this more in the IP part of the course
- Now, do we want to solve the problem all over again?
  - Seems like we could do some sort of "restart"
  - However, adding this constraint will make the problem infeasible

Graphical Depiction of the Cut





Adding the Constraint to the Tableau

- Here's the LP tableau at optimality (via LINDO):

Row	z	x1	x2	s1	s2	RHS	BV
0	1	0	0	-2/5	-9/5	44/5	z
1		1	0	-2/5	1/5	4/5	x1
2		0	1	1/5	-3/5	8/5	x2

note – since this is a min problem

- Here's the new tableau with the constraint, slack s_3:

Row	z	x1	x2	s1	s2	s3	RHS	BV
0	1	0	0	-2/5	-9/5	0	44/5	z
1		1	0	-2/5	1/5	0	4/5	x1
2		0	1	1/5	-3/5	0	8/5	x2
3		0	0	-1/5	-2/5	1	-3/5	s3

- Is this primal feasible? Dual feasible?

Introduction to Dual Simplex

- The tableau *is* dual feasible
  - Adding a row to the dual is the same as adding a column to the primal
  - **Can you make the primal infeasible by adding more variables?**
- Leads to an alternative scheme, called *dual simplex*
  - Discovered by C. E. Lemke in 1954 (Lemke was George Dantzig's first doctoral student)
  - Iterates among *dual* feasible solutions in a *primal* tableau
  - Improvements in dual simplex are responsible in dramatic improvements in LP solve times in the 1990's
  - More importantly, a key method for adding constraints in integer programming

Pivoting in Dual Simplex

- This method is a “transpose” of primal simplex
  - The pivot row is the most *negative* RHS
  - We only pivot on columns with *negative* coefficients
  - The ratio test is computed using the *objective function row;* take the ratio with the smallest ***absolute value***
- Example:

Row	z	x1	x2	s1	s2	s3	RHS	BV
0	1	0	0	-2/5	-9/5	0	44/5	z
1		1	0	-2/5	1/5	0	4/5	x1
2		0	1	1/5	-3/5	0	8/5	x2
3		0	0	-1/5	-2/5	1	-3/5	s3

ratio = 2

ratio = 9/2

Dual Simplex Termination

- Dual simplex finishes when the tableau is *primal feasible*
  - Recall that we started, and stay, dual feasible
  - If both primal and dual are feasible, then where are we?
- Row operations are exactly the same in dual simplex
  - Once you pick a pivot element, you get a 1 there, and 0's in the rest of the column
  - Here's the tableau after the pivot:

Row	z	x1	x2	s1	s2	s3	RHS	BV
0	1	0	0	0	-1	2	10	z
1		1	0	0	1	-2	2	x1
2		0	1	0	-1	1	1	x2
3		0	0	1	2	-5	3	s1

- It's optimal, **and integer**

Dual Simplex as a Solution Method

- Consider the starting tableau for the same problem:

Row	z	x1	x2	s1	s2	RHS	BV
0	1	-3	-4	0	0	0	z
1		3	1	-1	0	4	x1
2		1	2	0	-1	4	x2

- We can't do primal simplex; no BFS, need Phase I

Row	z	x1	x2	s1	s2	RHS	BV
0	1	-3	-4	0	0	0	z
1		-3	-1	1	0	-4	x1
2		-1	-2	0	1	-4	x2

- This *equivalent* tableau, however, is ***dual feasible***; we can do dual simplex immediately

The Pivots

Row	z	x1	x2	s1	s2	RHS	BV
0	1	-3	-4	0	0	0	z
1		-3	-1	1	0	-4	s1
2		-1	-2	0	1	-4	s2

Row	z	x1	x2	s1	s2	RHS	BV
0	1	0	-3	-1	0	4	z
1		1	1/3	-1/3	0	4/3	x1
2		0	-5/3	-1/3	1	-8/3	s2

Row	z	x1	x2	s1	s2	RHS	BV
0	1	0	0	-2/5	-9/5	44/5	z
1		1	0	-2/5	1/5	4/5	x1
2		0	1	1/5	-3/5	8/5	x2

Comprehensive Example

- This is a small problem
- Intended to show the entire process
  - Initial problem statement
  - First formulations
  - First solutions
  - Reformulations and modifications
  - Subsequent solutions
  - Sensitivity analysis
- Typical stumbling blocks

The Situation

- A group of investors wants to start a small passenger airline operation
  - The area they're targeting is currently only served by inconvenient hub-and-spoke routes
  - They believe they can compete and not get crushed in a price war; specialize in charters
  - They have a route structure and can lease various aircraft
  - The need to schedule their routes
- They call you in to assist
  - After some conversation, you believe you can model the problem
  - You're sent off to gather relevant data

Initial Information

- You meet with others involved in the new company
  - Most are irritated an outsider has been brought in
  - Cooperation is grudging; management has to threaten one group (the market forecasters) to get them to talk to you
- Here’s the initial information
  - The airline wants to cover 5 routes
  - They have a forecast for demand on each route
  - They have leased 4 different aircraft types
  - Tentative operating costs ($/ac/route) are available for each aircraft type
  - The pax capacity of each aircraft is known

What's the Objective and the Constraints?

- Minimize overall cost?
  - Only costs we have are operating (marginal) costs
  - Company claims to have fixed costs in hand, so you don't have to worry about them
- Other questions you might ask
  - Does it matter whether we have multiple aircraft types? (no, all lease, with contract maintenance)
  - Can we get different aircraft configurations? (No)
  - Are there limits on the number of aircraft available (No, they don't think so)
  - Does all demand have to be met? (Yes)
  - Is there a maximum operating cost? (No … but they hadn't considered this yet)

Your Initial Formulation

- Determine the aircraft mix that:
  - Minimizes total operating cost, and
  - Covers all demand
- Management agrees
- Indicies:
  - ***a***: aircraft types
  - ***r***: routes
- Data
  - $\boldsymbol{DEMAND_r}$ = passengers flying route ***r*** per month (100's)
  - $\boldsymbol{COST_{ar}}$ = \$1K/month to operate aircraft type ***a*** on route ***r***
  - $\boldsymbol{CAP_a}$ = maximum *monthly* capacity of aircraft type ***a*** (100's)

Formulation, cont'd

- Variables
  - $\boldsymbol{aca_{ar}}$ = # of aircraft a assigned to route r per day
- Objective and Constraints

$$\min z = \sum_{a,r} COST_{ar} * aca_{ar}$$

subject to

$$\sum_{a} CAP_{ar} * aca_{ar} \geq DEMAND_r \text{ for all } r$$

$$aca_{ar} \geq 0 \text{ for all } a,r$$

MPL Code

```
INDEX
    a := (ac1, ac2, ac3, ac4);      { aircraft types }
    r := (r1, r2, r3, r4, r5);      { routes }

DATA
    COST[a,r]  := (18,21,18,16,10,    { cost of aircraft a on route r, $1k/month }
                    0,15,16,14,9,
                    0,10,0,9,6,
                    17,16,17,15,10);  { NOTE: 0 cost means can't fly that route! }

    CAP[a,r]   := (16,15,28,23,81,    { capacity of aircraft a on route r, 100’s/month }
                   0,10,14,15,57,
                   0,5,0,7,29,
                   9,11,22,17,55);     { NOTE: 0 capacity means can't fly that route! }

    DEMAND[r]   := (253,120,180,80,600); { demand per month (100’s) on route r }

DECISION VARIABLES
    aca[a,r];    { number of aircraft a flying on route r }

MODEL
    MIN  totexpcost = SUM(a,r: COST[a,r]*aca[a,r]);

SUBJECT TO
    demreq[r]:  { demand constraints }

      SUM(a: CAP[a,r]*aca[a,r])  >  DEMAND[r] ;

END
```

Initial Solution

- Initial solution: use nothing but aircraft type 1
  - Optimal cost: $698K/month
  - Assignment data:

```
VARIABLE aca[a,r] :
   a    r            Activity     Reduced Cost
 -------------------------------------------
   ac1  r1            15.8125           0.0000
   ac1  r2             8.0000           0.0000
   ac1  r3             6.4286           0.0000
   ac1  r4             3.4783           0.0000
   ac1  r5             7.4074           0.0000
```

- What do you think the optimal integer solution is? Why?
- Change MPL code as follows to see:

```
INTEGER VARIABLES
    aca[a,r];    { number of aircraft a flying on route r }
```

Integer Solution and First Revisions

- The best integer solution is NOT to use all AC 1:
  - Optimal cost: $720K/month
  - Aircraft assignments:

a	r	Activity	Reduced Cost
ac1	r1	16.0000	18.0000
ac1	r2	8.0000	0.0000
ac1	r3	6.0000	18.0000
ac1	r4	2.0000	-4.2941
ac1	r5	6.0000	-2.7895
ac2	r3	1.0000	16.0000
ac2	r5	2.0000	0.0000
ac4	r4	2.0000	0.0000

- You present this to management
  - They say “we forgot; we can’t get that many of AC 1”
  - It turns out there’s limits on availability of *all* the aircraft types

Model Adjustments

- Data
  - ***$AVAIL_a$*** = number of aircraft a available
- New Model

$$\min\ z = \sum_{a,r} COST_{ar} * aca_{ar}$$

subject to

$$\sum_{a} CAP_{ar} * aca_{ar} \geq DEMAND_r \text{ for all } r$$

$$\sum_{r} aca_{ar} \leq AVAIL_a \text{ for all } a$$

$$aca_{ar} \geq 0 \text{ for all } a,r$$

First Model Death

- Here's the MPL changes:

```
AVAIL[a]    := (10,19,25,15);  { aircraft availability }

acavail[a]: { aircraft availability }
      SUM(r: aca[a,r]) < AVAIL[a];
```

- You run the model, and MPL says "integer infeasible"
  - What happened?
  - Change it back to an LP, see if it solves; it's *still* infeasible
  - Now what?
- Solve a different problem
  - Minimize the unmet demand, given the aircraft availability
  - See if you can figure out what combinations are causing trouble

The Next Model - Where Are We Short?

- Here's the new formulation; minimize unmet demand

$$\min\ z = \sum_r unmet_r$$

subject to

$$\sum_a \left(CAP_{ar} * aca_{ar}\right) + unmet_r \geq DEMAND_r \text{ for all } r$$

$$\sum_r aca_{ar} \leq AVAIL_a \text{ for all } a$$

$$aca_{ar} \geq 0 \text{ for all } a,r$$

$$unmet_r \geq 0 \text{ for all } r$$

Is this right? Why?

The Answers

- LP results - close, but can't satisfy Route 1

VARIABLE unmet[r] :

r	Activity	Reduced Cost
r1	9.7476	0.0000
r2	0.0000	0.4273
r3	0.0000	0.5909
r4	0.0000	0.6182
r5	0.0000	0.9013

CONSTRAINT acavail[a] :

a	Slack	Shadow Price
ac1	0.0000	-16.0000
ac2	0.0000	-5.7273
ac3	0.0000	-2.8636
ac4	0.0000	-9.0000

- Which aircraft type do we probably want more of?
- Note that the integer answer is somewhat worse:

VARIABLE unmet[r] :

r	Activity	Reduced Cost
r1	12.0000	0.0000
r2	0.0000	0.0000
r3	0.0000	0.2857
r4	3.0000	0.0000
r5	0.0000	0.7586

Negotiations with the Customer

- Marketing group is upset; claims answer is wrong
- They show the following table:

Aircraft Capacity					
Route	**AC 1**	**AC 2**	**AC 3**	**AC 4**	**Max**
1	16	0	0	9	295
2	15	10	5	11	630
3	28	14	0	22	876
4	23	15	7	17	945
5	81	57	29	55	3443
AC Avail	**10**	**19**	**25**	**15**	

- How would you argue your way out of this?
  - But, suppose you win
  - Management says, “get with marketing and figure this out”

Adding a Bumping Cost

- Marketing says, “we can bump people at a price”
  - **Data:** $\boldsymbol{BPCOST_r}$ = \$K lost per 100 passengers bumped on route $\boldsymbol{r}$
  - **Variable:** $\boldsymbol{bumped_r}$ = passengers bumped on route $\boldsymbol{r}$ (100’s)
- New model:

$$\min z = \sum_{a,r} COST_{ar} * aca_{ar} + \sum_{r} BPCOST_r * bumped_r$$

subject to

$$\sum_{a} CAP_{ar} * aca_{ar} + bumped_r \geq DEMAND_r \text{ for all } r$$

$$\sum_{r} aca_{ar} \leq AVAIL_a \text{ for all } a$$

$$aca_{ar} \geq 0 \text{ for all } a,r \text{ ; } bumped_r \geq 0 \text{ for all } r$$

The New Solution

- LP solution: z = $999K/month

VARIABLE bumped[r] :

r	Activity	Reduced Cost
r1	0.0000	1.3016
r2	0.0000	6.4000
r3	0.0000	2.2143
r4	0.0000	2.6667
r5	98.7143	0.0000

CONSTRAINT acavail[a] :

a	Slack	Shadow Price
ac1	0.0000	-169.1746
ac2	0.0000	-51.0000
ac3	0.0000	-23.0000
ac4	0.0000	-88.2857

- Integer solution: z = $1012K/month

VARIABLE bumped[r] :

r	Activity	Reduced Cost
r1	3.0000	0.0000
r2	0.0000	6.4000
r3	0.0000	2.2143
r4	0.0000	2.4286
r5	78.0000	0.0000

CONSTRAINT acavail[a] :

a	Slack	Shadow Price
ac1	0.0000	-190.0000
ac2	0.0000	-51.0000
ac3	0.0000	-23.0000
ac4	0.0000	-88.2857

The Management Responds

- Leadership doesn't like the answer
  - Almost all the bumping occurs in route 5
  - Wants the risk of bumping spread out more evenly across routes
  - Now what?
- First, check for multiple optima in the solution
  - May be an alternative that is cost optimal, but spreads out bumps
  - But, there are none in the LP solution
  - This means that spreading out bumping will cost more
- Note, however, that this is based on *expected* demand
  - Marketing says forecasts probably good to within 5%
  - Implies that total costs have about 5% accuracy as well
  - This is how we will try to spread out bumping

New Model to Spread Out Bumping

- Previous objective function now becomes a constraint
- We add a new variable, ***maxbump***
- Here's the new model:

NOTE

$$\min z = maxbump$$

subject to

$$\sum_{a,r} COST_{ar} * aca_{ar} + \sum_{r} BPCOST_{r} * bumped_{r} \leq 1.05 * z_{original}$$

$$maxbump \geq bumped_{r} \text{ for all } r$$

$$\sum_{a} CAP_{ar} * aca_{ar} + bumped_{r} \geq DEMAND_{r} \text{ for all } r$$

$$\sum_{r} aca_{ar} \leq AVAIL_{a} \text{ for all } a$$

$$aca_{ar} \geq 0 \text{ for all } a,r \, ; bumped_{r} \geq 0 \text{ for all } r$$

And, What Happens?

- This solution does indeed spread out bumping:

```
VARIABLE bumped[r] :

 r            Activity     Reduced Cost
---------------------------------------
 r1             3.9969           0.0000
 r2             3.8324           0.0000
 r3             3.9969           0.0000
 r4             3.9969           0.0000
 r5             3.9969           0.0000
---------------------------------------
```

- This does not really make things equitable
  - Demand differs on each route
  - Management wants an equal chance of bumping on each route
  - Need to recast ***maxbump*** as a proportion of route demand

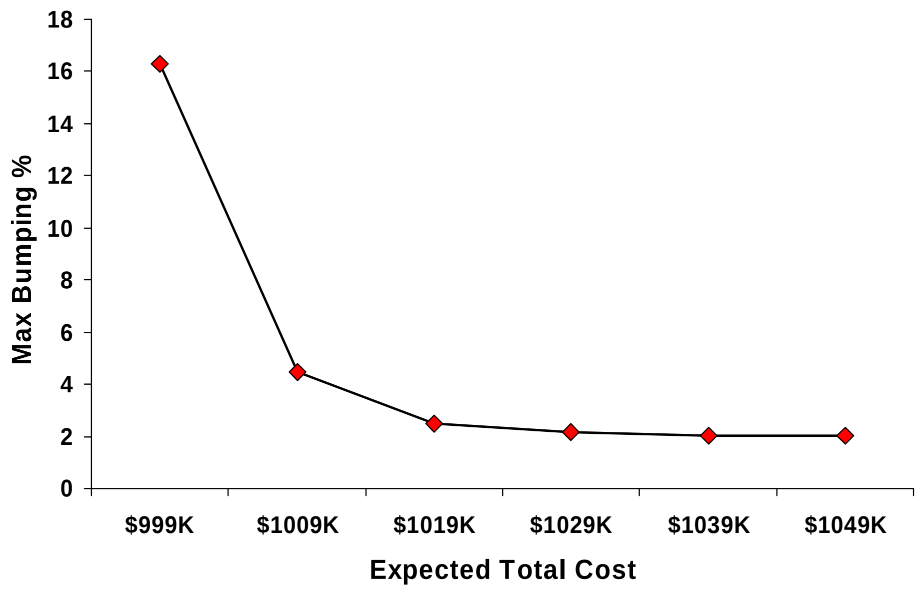
Bumping As a Proportion

- This solution has the optimal ***maxbump*** at 2.01%
- New bumping results:

```
VARIABLE bumped[r] :

 r              Activity     Reduced Cost
 ---------------------------------------
 r1              5.0832           0.0000
 r2              2.4110           0.0000
 r3              3.6165           0.0000
 r4              1.6073           0.0000
 r5             12.0551           0.0000
 ---------------------------------------
```

- Here's how it varies by total cost:



Demand Scrutiny

- However, this whole exercise causes scrutiny of demand forecast
- Management to marketing: “Where the #$%^@&!! did this come from?”
- Marketing digs through the files, comes up with the following spreadsheet data

The Original Demand Data

- Here's where the expected demand was derived from:

DEMAND

	Demand State				
Route	**1**	**2**	**3**	**4**	**5**
1	200	220	250	270	300
2	50	150			
3	140	160	180	200	220
4	10	50	80	100	340
5	580	600	620		

LIKELIHOOD

	Demand State				
Route	**1**	**2**	**3**	**4**	**5**
1	0.2	0.05	0.35	0.2	0.2
2	0.3	0.7			
3	0.1	0.2	0.4	0.2	0.1
4	0.2	0.2	0.3	0.2	0.1
5	0.1	0.8	0.1		

- Looks like it's time for a recourse model

Extracting Scenarios

- Note that this data is by route
  - The ***joint*** distribution of demand is unclear
  - Seems reasonable, though, that if demand is high on one route, it is probably also high on another
- We decide to use 6 scenarios:

	scenario					
	s1	**s2**	**s3**	**s4**	**s5**	**s6**
probability	0.2	0.2	0.2	0.2	0.1	0.1
route 1	200	243	250	270	300	300
route 2	50	100	150	150	150	150
route 3	150	170	180	190	200	220
route 4	10	50	80	90	100	340
route 5	590	600	600	600	600	620

The New Scenario Model

- Go back to minimizing cost, but add:
  - Index s = scenario
  - Data ***SPROB$_s$*** = probability of scenario ***s***
  - Add ***s*** index to demand data and bumping variables
- New model

$$\min z = \sum_{a,r} COST_{ar} * aca_{ar} + \sum_{r,s} SPROB_s * BPCOST_r * bumped_{rs}$$

subject to

$$\sum_a CAP_{ar} * aca_{ar} + bumped_{rs} \geq DEMAND_{rs} \text{ for all } r, s$$

$$\sum_r aca_{ar} \leq AVAIL_a \text{ for all } a$$

$$aca_{ar} \geq 0 \text{ for all } a, r \,;\, bumped_{rs} \geq 0 \text{ for all } r, s$$

As You Would Expect ...

- This answer is nowhere near as rosy
  - Total expected cost: $1562K
  - Operating cost: $887K
  - Bumping cost: $678K
- One route/scenario combo has 26,000 pax/month unmet demand
- Conversation ensues
  - **First question:** what if the route data is all independent?
  - **Second question:** If the 6-scenario model is valid, what's the minimum number of aircraft needed to ensure a less than 10% chance of bumping on any route?

Assume the Route Demands are Independent

- Model mods:
  - Index ***d*** = demand state (1-5)
  - Data $\boldsymbol{DPROB_{rd}}$ = probability of demand state ***d*** on route ***r***
  - Data $\boldsymbol{DDEM_{rd}}$ = demand on route ***r*** in demand state ***d***
  - Variable $\boldsymbol{bumped_{rd}}$ = number bumped from route ***r*** in demand state ***d***
- New Model

$$\min z = \sum_{a,r} COST_{ar} * aca_{ar} + \sum_{r,d} DPROB_{rd} * BPCOST_r * bumped_{rd}$$

subject to

$$\sum_a CAP_{ar} * aca_{ar} + bumped_{rd} \geq DDEM_{rd} \text{ for all } r, d$$

$$\sum_r aca_{ar} \leq AVAIL_a \text{ for all } a$$

$$aca_{ar} \geq 0 \text{ for all } a,r \,;\, bumped_{rd} \geq 0 \text{ for all } r, d$$

Answer to the Independent Demand Case

- Total expected cost: $1566K
  - $883K operating cost
  - $683K bumping cost
  - Bumping statistics similar to scenario case
  - Integer answer: $1580K (very similar)
- Interesting result: total expected cost is slightly *higher* than the scenario case

Homework

- Answer the second question
  - Formulate and solve in MPL the case that minimizes the number of aircraft required to get less than 10% bumping for *any* route and scenario
    - Turn in separate formulation (written out, NOT MPL code)
    - Provide MPL code for new model
  - Also, investigate sensitivity of the solution for the range 5-15%
    - Changes in total costs
    - Changes in optimal fleet mixes
  - I have provided MPL code for the first question; work from there
- Also:
  - p. 335: 2a