Dakota Problem

	Desk	Table	Chair	

$$z = 60x_1 + 30x_2 + 20x_3 \rightarrow \max$$

y_1	$8x_1 + 6x_2 + x_3 \le 48$	*Lumber*
y_2	$4x_1 + 2x_2 + 1.5x_3 \le 20$	*Finishing*
y_3	$2x_1 + 1.5x_2 + 0.5x_3 \le 8$	*Carpentry*

$$x_i \ge 0$$

$$w = 48y_1 + 20y_2 + 8y_3 \rightarrow \min$$

$$8y_1 + 4y_2 + 2y_3 \ge 60$$

$$6y_1 + 2y_2 + 1.5y_3 \ge 30$$

$$y_1 + 1.5y_2 + 0.5y_3 \ge 20$$

$$y_i \ge 0$$

Weak Duality

x-primal feasible, y-dual feasible

$$Ax \le b \qquad A^T y \ge c$$

$$x \ge 0 \qquad y \ge 0$$

then

$$(c,x) \le (b,y)$$

$$(y, b-Ax) \ge 0 \Rightarrow (y,b) \ge (y,Ax)$$

$$(A^T y - c, x) \ge 0 \Rightarrow (A^T y, x) \ge (c,x)$$

$$(A^T y, x) \equiv (y, Ax)$$

$$(y,b) \ge (y,Ax) = (A^T y, x) \ge (c,x)$$

First Duality Theorem

Theorem 1) If one of the two dual LP problems has a solution the other has a solution and the op. v = i.e.

$$z^* = w^*$$

2) If one problem is unbounded the other is infeasible.

3) Both LP's might be infeasible.

Consider max z = (c,x) → max (c,x) + (0,s)

$$\begin{aligned} Ax &\le b & & Ax+s = b \\ & & \Rightarrow & \\ x &\ge 0 & & x \ge 0,\ s \ge 0 \end{aligned}$$

$$\begin{array}{c|ccccc|c} & x_B & x_N & & & & \\ & c_B & c_N & & 0 & & \\ \hline & & & 1 & 0 & 0 & \\ y & B & N & 0 & 1 & 0 & b \\ & & & 0 & 0 & 1 & \\ \hline \end{array}$$

$$y = c_B B^{-1} \Rightarrow yB = c_B,\ yB - c_B = 0$$

$$x_B = B^{-1}b, \quad z = (c_B, x_B) + (c_N, x_N)$$

$$= c_B B^{-1} b = (y, b) = w$$

$$\Delta_j = (y, A_j) - c_j \ge 0, \quad (y, e_i) = y_i \ge 0,$$

$$i = 1, \dots, m \qquad j = m+1, \dots, n$$

$$z = (c, x) = (y, b) = w$$

$$Ax \le c \qquad yB = c_{\mathbf{B}}$$

$$\Rightarrow yA \ge c$$

$$x \ge 0 \qquad yN \ge c_N \qquad \Downarrow$$

$$A^T y - c \ge 0$$

$$y \ge 0$$

x – *primal feasible*

$$z = w$$

y – *dual feasible*

$$\Downarrow$$

$$x = x^*, \quad y = y^*$$

Unbounded

	$c_j > 0$	
y_1	$a_{1j} \le 0$	b_1
y_i	$a_{ij} \le 0$	b_i
y_m	$a_{mj} \le 0$	b_m
	$\Delta_j < 0$	

$$a_{ij} \le 0, \qquad a_{mj} \le 0$$

$$x_j A_j \le 0 \quad \textit{for any} \quad x_j > 0$$

$$(y, A_j) \le 0, \quad \Delta_j = (y, A_j) - c_j < 0$$

$$(y, A_j) \ge c_j$$

Both Primal & Dual Infeasible

	x_1	x_2		
y_1	-2	2	-3	$-2x_1+2x_2 \leq -3$
y_2	2	-2	1	$2x_1-2x_2 \leq 1$
	3	-1		$x_1 \geq 0, x_2 \geq 0$

$-2y_1 + 2y_2 \geq 3$

$2y_1 - 2y_2 \geq -1$

$y_1 \geq 0, y_2 \geq 0$

Primal $3x_1 - x_2 \to \max$ Dual $-3y_1 + y_2 \to \min$

Primal	Dual
$-2x_1+2x_2 \leq -3$	$-2y_1 + 2y_2 \geq 3$
$2x_1-2x_2 \leq 1$	$2y_1 - 2y_2 \geq -1$
$x_1 \geq 0, x_2 \geq 0$	$y_1 \geq 0, y_2 \geq 0$

Second Duality Theorem

Complementarity Condition

$y_i \geq 0$, $(b\text{-}Ax)_i \geq 0$, $y_i\,(b\text{-}Ax)_i = 0$ $\quad i = 1,\ldots,m$

$x_j \geq 0$, $(A^Ty\text{-}c)_j \geq 0$, $x_j\,(A^Ty\text{-}c)_j = 0$ $\quad j = 1,\ldots,n$

Theorem: For a pair of feasible solutions (x,y) to be the optimal pair, it is necessary and sufficient to satisfy the complementarity condition

Name: Israel.

$n = 174$, $p = 316$,

it	gap	primal inf	dual inf	# of steps
0	1.05e+10	1.21e+06	1.78e+04	0
1	6.40e+00	2.77e-09	7.53e-10	20
2	7.364041e-02	1.0729e-07	0.00e+00	14
3	7.497715e-07	4.2011e-12	0.00e+00	6
4	1.628188e-10	1.7764e-15	0.00e+00	3
Total number of Newton's steps				43

Name: AGG.

$n = 488$, $p = 615$,

it	gap	primal inf	dual inf	# of steps
0	4.29e+11	3.78e+07	2.49e+04	0
1	1.23e+00	3.01e-06	7.16e-10	19
2	2.942798e-04	2.7691e-10	0.00e+00	3
3	3.949872e-09	2.8421e-14	0.00e+00	3
Total number of Newton's steps				25

Name: AGG2.

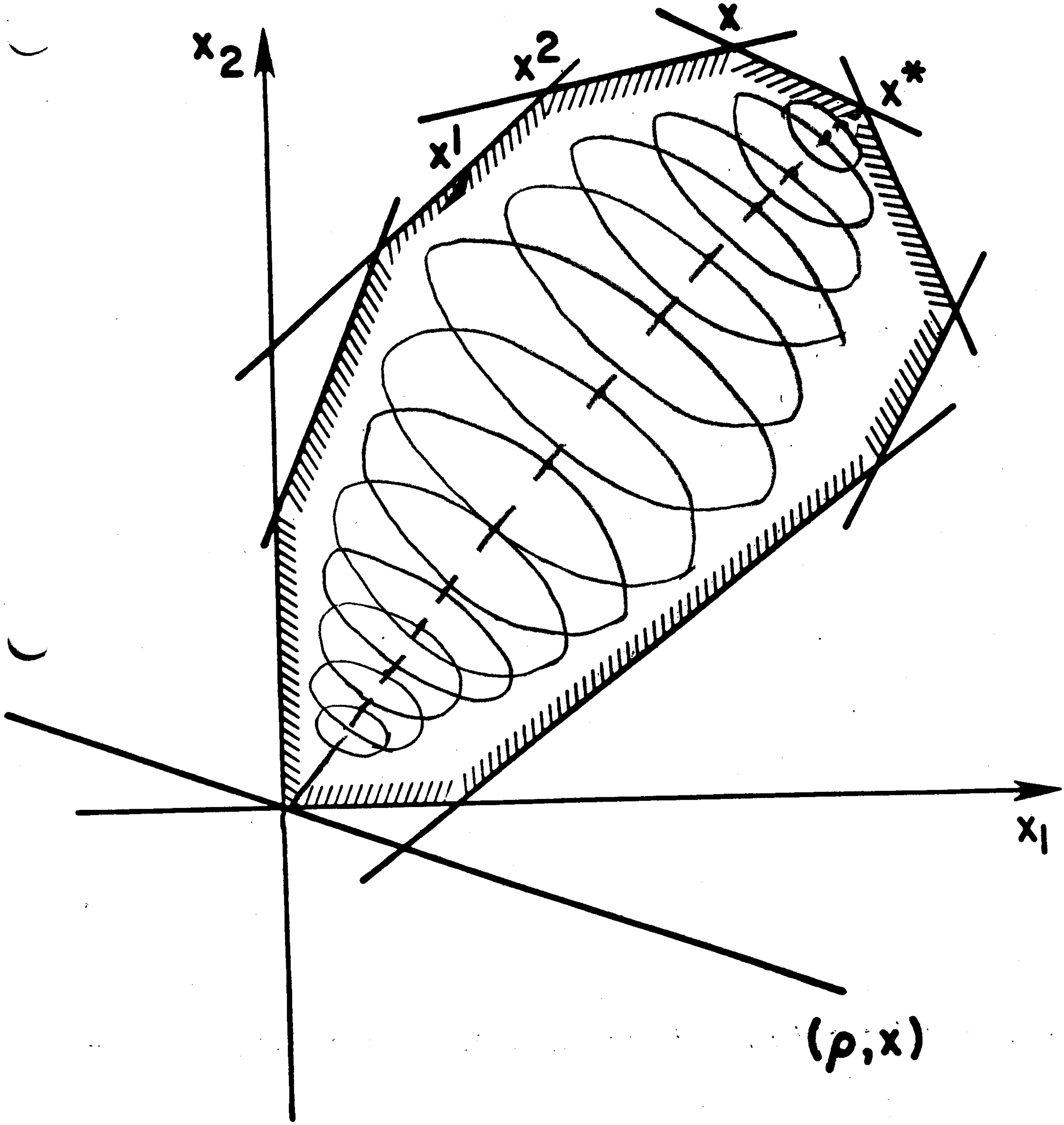
$n = 516$, $p = 758$,

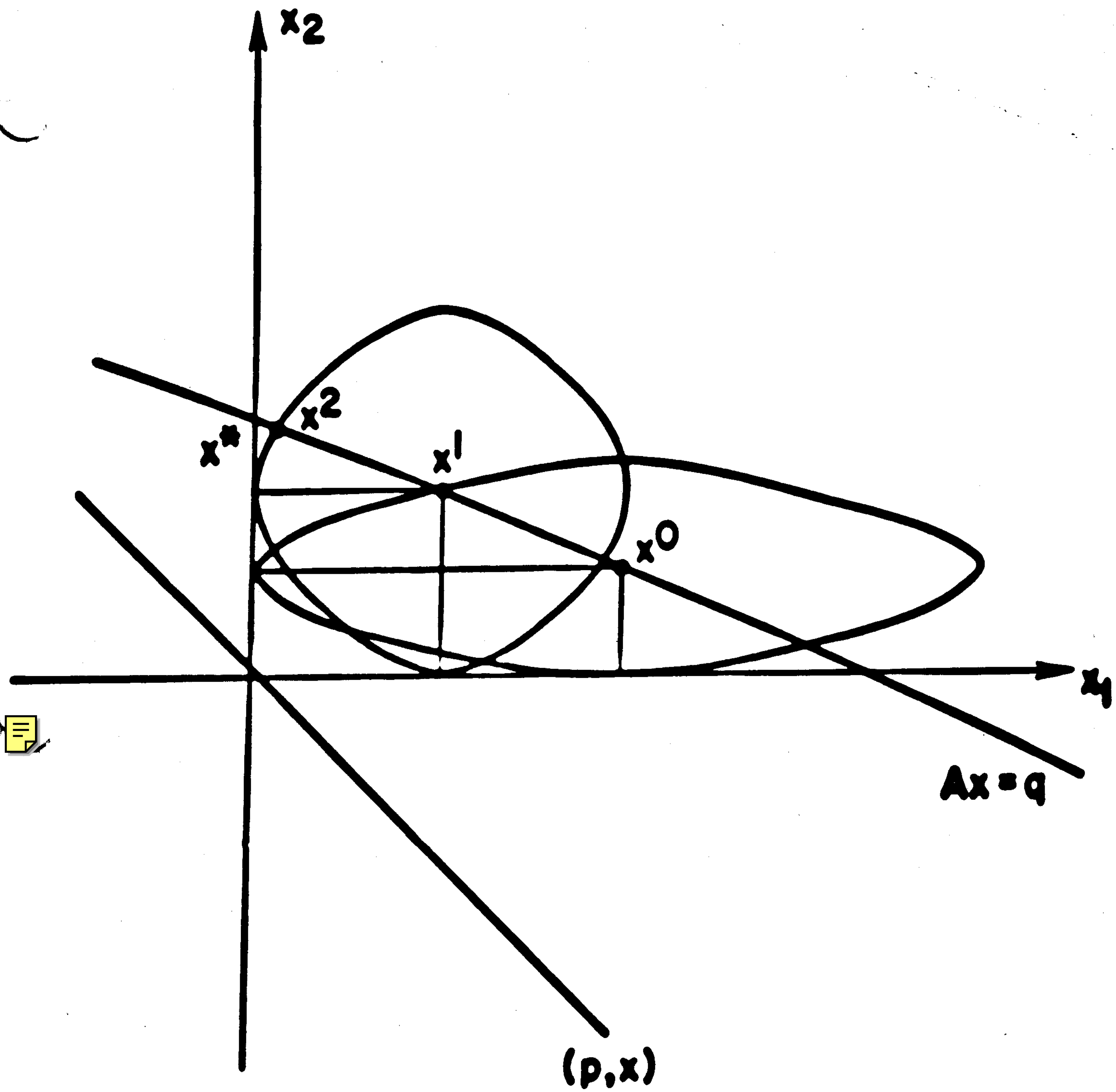
it	gap	primal inf	dual inf	# of steps
0	6.93e+10	7.41e+06	2.08e+04	0
1	6.07e+00	4.39e-07	5.41e-10	16
2	1.422620e-03	2.2625e-09	0.00e+00	3
3	2.630272e-10	7.1054e-15	0.00e+00	3
Total number of Newton's steps				25

Name: BNL1.

$n = 516$, $p = 758$,

it	gap	primal inf	dual inf	# of steps
0	3.98e+06	1.83e+04	8.41e+02	0
1	9.47e-05	7.13e-09	3.81e-12	25
2	2.645905e-07	3.8801e-10	0.00e+00	5
3	2.025197e-12	4.5938e-13	0.00e+00	4
Total number of Newton's steps				34


x2
x2
x
x*
x1
x1
(p,x)



$$(p, x^{s+1} - x^*) \le \left(1 - \frac{\alpha}{\sqrt{n - m + \varepsilon_s}}\right)(p, x^s - x^*), \quad \varepsilon_s \to 0$$

I.I. Dikin 1967