He was born Neumann Janos on December 28, 1903, in Budapest, the capital of his native Hungary. He died John von Neumann on February 8, 1957, of a tragically early cancer in Washington, D.C., the capital of his adopted United States. To friends and even acquaintances in America he was always known as "Johnny," as in Hungary he had been "Jancsi."

He was a prodigious child and a prodigious student, and through his brief fifty-three years grew steadily more prodigious. The most startling young innovator among the pure mathematicians of the 1920s, he surged on to leave his mark on theoretical physics and then on dramatically applied physics, on decision theory, on meteorology, on biology, on economics, on deterrence to war—and eventually became, more than any other individual, the creator of the modern digital computer and the most farsighted of those who put it to early use. He marked up nearly all his achievements while he was mainly engaged in something else.

In each century there are a handful of people who, grappling with problems in their lonely brains, write a few equations on a few blackboards, and the world changes. Johnny was among the most consistently effective of the mathematicians in our century—which possibly means in any century hitherto, because we can now do such extraordinary things so quickly once these men have worked out their sums.

If Johnny had not lived, the development of America's nuclear, thermonuclear, and certainly missile-borne deterrent would have been slower, maybe fatally so. Without him, the computer revolution would not yet have reached its present foothills, from which so many new roads will go. In his last decade the often-terrifying clarity of his mind was at the service of the Truman and especially Eisenhower administrations.

Duality

$$\max z = c_1 x_1 + \ldots + c_n x_n$$

$$\begin{array}{ll} y_1 & a_{11}x_1 + \ldots + a_{1n}x_n \le b_1 \\ & \ldots\ldots\ldots\ldots\ldots \\ y_i & a_{i1}x_1 + \ldots + a_{in}x_n \le b_i \\ & \ldots\ldots\ldots\ldots\ldots \\ y_m & a_{m1}x_1 + \ldots + a_{mn}x_n \le b_m \end{array}$$

$$x_1 \ge 0, \ldots, x_n \ge 0$$

$$\min w = b_1 y_1 + \ldots + b_m y_m$$

$$a_{11}y_1 + \ldots + a_{i1}y_i + \ldots + a_{m1}y_m \ge c_1$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{1n}y_1 + \ldots + a_{in}y_i + \ldots + a_{mn}y_m \ge c_n$$

$$y_1 \ge 0, \ldots, y_m \ge 0$$

Rules

1. max $\Rightarrow$ min

2. c $\rightarrow$ r.h.s., b $\rightarrow$ o.f.

3. $\leq \Rightarrow \geq 0$

4. $A \rightarrow A^T$

5. The num. of primal var. = num. of dual constr.

   The num. of dual var. = num. of primal constr.

$z = (c,x) \rightarrow \max$	$w = (b,y) \rightarrow \min$
$Ax \leq b$	$A^T y \geq c$
$x \geq 0$	$y \geq 0$

Dakota Problem

	Desk		Table		Chair	

$$z = 60x_1 + 30x_2 + 20x_3 \to \max$$

y_1 $\quad 8x_1 + 6x_2 + x_3 \le 48 \quad$ *Lumber*

y_2 $\quad 4x_1 + 2x_2 + 1.5x_3 \le 20 \quad$ *Finishing*

y_3 $\quad 2x_1 + 1.5x_2 + 0.5x_3 \le 8 \quad$ *Carpentry*

$$x_i \ge 0$$

$$w = 48y_1 + 20y_2 + 8y_3 \to \min$$

$$8y_1 + 4y_2 + 2y_3 \ge 60$$

$$6y_1 + 2y_2 + 1.5y_3 \ge 30$$

$$y_1 + 1.5y_2 + 0.5y_3 \ge 20$$

$$y_i \ge 0$$

Example

$$z = 3x_1 + 5x_2 + 7x_3 + x_4 \to \max$$

$$y_1 \qquad x_1 + 2x_2 + 3x_3 + 5x_4 \le 10$$

$$y_2 \qquad 2x_1 + 4x_2 + x_3 + x_4 \le 15$$

$$x_i \ge 0$$

$$w = 10y_1 + 15y_2$$

$$y_1 + 2y_2 \ge 3$$

$$2y_1 + 4y_2 \ge 5$$

$$3y_1 + y_2 \ge 7$$

$$5y_1 + y_2 \ge 1$$

$$y_1 \ge 0,\ y_2 \ge 0$$

Weak Duality

x-primal feasible, y-dual feasible

$$Ax \leq b \qquad A^T y \geq c$$

$$x \geq 0 \qquad y \geq 0$$

then

$$(c,x) \leq (b,y)$$

$$(y,b-Ax) \geq 0 \Rightarrow (y,b) \geq (y,Ax)$$

$$(A^T y-c,x) \geq 0 \Rightarrow (A^T y,x) \geq (c,x)$$

$$(A^T y,x) \equiv (y,Ax)$$

$$(y,b) \geq (y,Ax) = (A^T y,x) \geq (c,x)$$

First Duality Theorem

Theorem 1) If one of the two dual LP problems has a solution the other has a solution and the op. v = i.e.

$$z^* = w^*$$

2) If one problem is unbounded the other is infeasible.

3) Both LP's might be infeasible.

Consider max $z = (c,x) \rightarrow \max\ (c,x) + (0,s)$

$$\begin{array}{ccc} Ax \le b & & Ax+s = b \\ & \Rightarrow & \\ x \ge 0 & & x \ge 0,\ s \ge 0 \end{array}$$

$$\begin{array}{c|ccccc|c} & x_B & x_N & & & & \\ & c_B & c_N & & 0 & & \\ \hline & & & 1 & 0 & 0 & \\ y & B & N & 0 & 1 & 0 & b \\ & & & 0 & 0 & 1 & \\ \hline \end{array}$$

$$y = c_B^T B^{-1} \Rightarrow yB = c_B,\ y^T B - c_B^T = 0$$

$$x_B = B^{-1} b, \quad z = (c_B, x_B) + (c_N, x_N)$$

$$= c_B^T B^{-1} b = (y, b) = w$$

$$\Delta_j = (y, A_j) - c_j \ge 0, \quad (y, e_i) = y_i \ge 0,$$

$$i = 1, \ldots, m \quad j = m+1, \ldots, n$$

$$z = (c, x) = (y, b) = w$$

$$Ax \le c \qquad y^T B = c_B^T$$

$$\Rightarrow y^T A \ge c$$

$$x \ge 0 \qquad y^T N \ge c_N \qquad \Downarrow$$

$$A^T y - c \ge 0$$

$$y \ge 0$$

x – *primal feasible*

$$z = w$$

y – *dual feasible*

$$\Downarrow$$

$$x = x^*, \quad y = y^*$$

Unbounded

	$c_j > 0$	
y_1	$a_{1j} \le 0$	b_1
y_i	$a_{ij} \le 0$	b_i
y_m	$a_{mj} \le 0$	b_m
	$\Delta_j < 0$	

$$a_{ij} \le 0, \qquad a_{mj} \le 0$$

$$x_j A_j \le 0 \quad for \quad any \quad x_j > 0$$

$$(y, A_j) \le 0, \;\; \Delta_j = (y, A_j) - c_j < 0$$

$$(y, A_j) \ge c_j$$

Both Primal & Dual Infeasible

	x_1	x_2		
y_1	-2	2	-3	$-2x_1+2x_2 \le -3$
y_2	2	-2	1	$2x_1-2x_2 \le 1$
	3	-1		$x_1 \ge 0, x_2 \ge 0$

$-2y_1 + 2y_2 \ge 3$

$2y_1 - 2y_2 \ge -1$

$y_1 \ge 0, y_2 \ge 0$

Primal $3x_1 - x_2 \to \max$ Dual $-3y_1 + y_2 \to \min$

$-2x_1+2x_2 \le -3$	$-2y_1 + 2y_2 \ge 3$
$2x_1-2x_2 \le 1$	$2y_1 - 2y_2 \ge -1$
$x_1 \ge 0, x_2 \ge 0$	$y_1 \ge 0, y_2 \ge 0$

Second Duality Theorem

Complementarity Condition

$y_i \geq 0,\ (b-Ax)_i \geq 0,\ y_i\,(b-Ax)_i = 0 \qquad i = 1,\ldots,m$

$x_j \geq 0,\ (A^T y-c)_j \geq 0,\ x_j\,(A^T y-c)_j = 0 \qquad j = 1,\ldots,n$

Theorem: For a pair of feasible solutions (x,y) to be the optimal pair, it is necessary and sufficient to satisfy the complementarity condition

Necessary

$$b - Ax \geq 0,\ \ y \geq 0,\ \ A^T y - c \geq 0,\ \ x \geq 0$$

$$(b, y) \geq (Ax, y) = (A^T y, x) \geq (c, x)$$

If $x = x^*,\ \ y = y^*$ *then* $(b, y^*) = (c, x^*)$

and

$$(b, y^*) = (Ax^*, y^*) = (A^T y^*, x^*) = (c, x^*)$$

$$(b - Ax^*, y^*) = 0,\ \ x^*(A^T y^* - c) = 0$$

Sufficient

$$y_i^*(b - Ax^*)_i = 0,\quad x_j^*(A^T y^* - c)_j = 0$$

If the complementarity cond. are true

then $(b, y) = (c, x) \Rightarrow x = x^*,\ \ y = y^*$

$$(Ax^*)_i - b_i < 0 \Rightarrow y_i^* = 0$$

$$y_i^* > 0 \Rightarrow (Ax^*)_i - b_i = 0$$

$$(A^T y^* - c)_j > 0 \Rightarrow (A_j, y^*) > c_j \Rightarrow x_j^* = 0$$

$$x_j^* > 0 \Rightarrow (A^T y^* - c)_j = 0$$

$$z = c^T x \to \max$$
$$A_1 x \le b_1$$
$$A_2 x = b_2$$
$$x \ge 0$$

$$w = b_1^T y_1 + b_2^T y_2 \to \min$$
$$A_1^T y_1 + A_2^T y_2 \ge c$$
$$y_1 \ge 0,\ \ y_2 - unconstr.$$

$$\max\ z = (c, x)$$

$$y_1 \qquad A_1 x \le b_1$$

$$y_2 \qquad A_2 x = b_2$$

$$y_3 \qquad A_3 x \ge b_3 \Leftrightarrow -A_3 x \le -b_3$$

$$x \ge 0$$

$$\min\ w = (b_1, y_1) + (b_2, y_2) - (b_3, y_3)$$

$$A_1^T y_1 + A_2^T y_2 - A_3^T y_3 \ge c$$

$$y_1 \ge 0,\ \ y_2 - UNR.\ \ y_3 \ge 0$$

$$or\ \ \min\ w = (b_1, y_1) + (b_2, y_2) + (b_3, y_3)$$

$$A_1^T y_1 + A_2^T y_2 + A_3^T y_3 \ge c$$

$$y_1 \ge 0,\ y_2 - UNR,\ y_3 \le 0$$

y_1	A_{11}	A_{12}	$\leq$	b_1
y_2	A_{21}	A_{22}	$\not\leq$	b_2

$$x_1 \geq 0 \quad x_2 \; unr$$

$$z = c_1^T x_1 + c_2^T x_2 \rightarrow \max$$

$$A_{11} x_1 + A_{12} x_2 \leq b_1$$

$$A_{21} x_1 + A_{22} x_2 = b_2$$

$$x_1 \geq 0, \; x_2 \; unr$$

$$w = b_1^T y_1 + b_2^T y_2$$

$$A_{11}^T y_1 + A_{21}^T y_2 \geq c_1$$

$$A_{12}^T y_1 + A_{22}^T y_2 = c_2$$

$$y_1 \geq 0, \; y_2 \; unr$$

Example

$$z = 3x_1 + x_2 - x_3 \rightarrow \max$$

$$y_1 \quad 2x_1 - x_2 + 3x_3 \le 4$$

$$y_2 \quad -5x_1 + 2x_2 - 7x_3 \ge 5 \Rightarrow$$

$$5x_1 - 2x_2 + 7x_3 \le -5$$

$$y_3 \quad x_1 - 4x_2 + x_3 = 3$$

$$x_1 \ge 0, x_2 \ unr, x_3 \ge 0$$

$$w = 4y_1 - 5y_2 + 3y_3 \rightarrow \min$$

$$2y_1 + 5y_2 + y_3 \ge 3$$

$$-y_1 - 2y_2 - 4y_3 = 1$$

$$3y_1 + 7y_2 + y_3 \ge -1$$

$$y_1 \ge 0, y_2 \ge 0, y_3 \ unr$$

$$z = 4x_1 + x_2 + \tfrac{1}{4}x_3 \to \max$$
$$8x_1 + 3x_2 + x_3 \le 4$$
$$6x_1 + x_2 + x_3 \le 2$$
$$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0$$

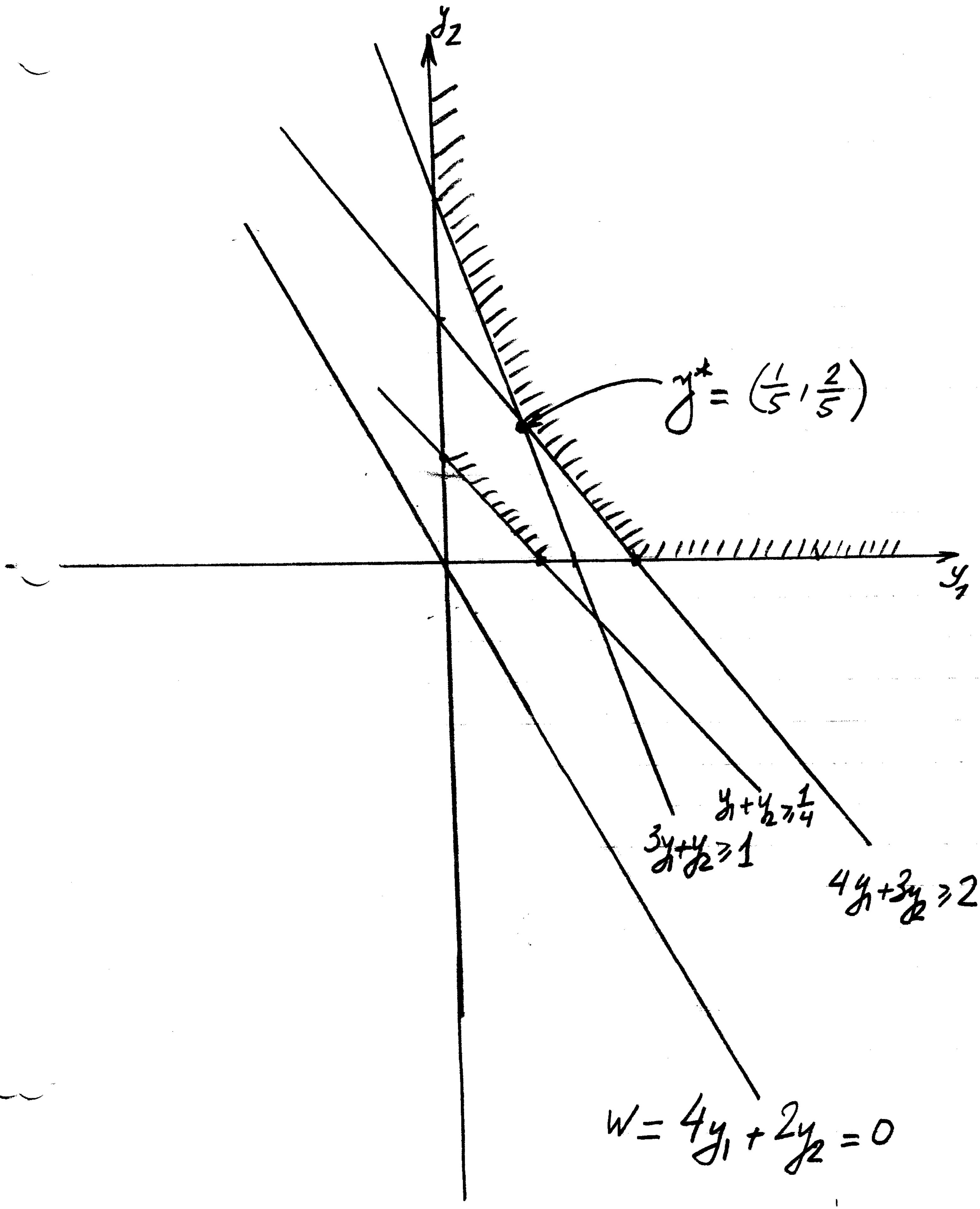
Dual $w = 4y_1 + 2y_2 \to \min$

$$8y_1 + 6y_2 \ge 4 \Rightarrow 4y_1 + 3y_2 \ge 2$$
$$3y_1 + y_2 \ge 1$$
$$y_1 + y_2 \ge \frac{1}{4}$$
$$y_1 \ge 0,\ y_2 \ge 0$$

$$\begin{aligned} 4y_1 + 3y_2 &= 2 \\ 3y_1 + y_2 &= 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 4y_1 + 3y_2 &= 2 \\ 9y_1 + 3y_2 &= 3 \end{aligned}$$

$$5y_1 = 1,\ y_1 = 1/5$$

$$4/5 + 3y_2 = 2 \quad \Rightarrow \quad y_2 = 2/5$$



y_2
y_1
$y^* = (\frac{1}{5}, \frac{2}{5})$
$y_1 + y_2 \geq \frac{1}{4}$
$3y_1 + y_2 \geq 1$
$4y_1 + 3y_2 \geq 2$
$w = 4y_1 + 2y_2 = 0$

$$\max \quad z = 5x_1 + 3x_2 + x_3$$

$$y_1 \qquad 2x_1 + x_2 + x_3 \le 6$$

$$y_2 \qquad x_1 + 2x_2 + x_3 \le 7$$

$$x_i \ge 0$$

Dual

$$\min \quad w = 6y_1 + 7y_2$$

$$2y_1 + y_2 \ge 5$$

$$y_1 + 2y_2 \ge 3$$

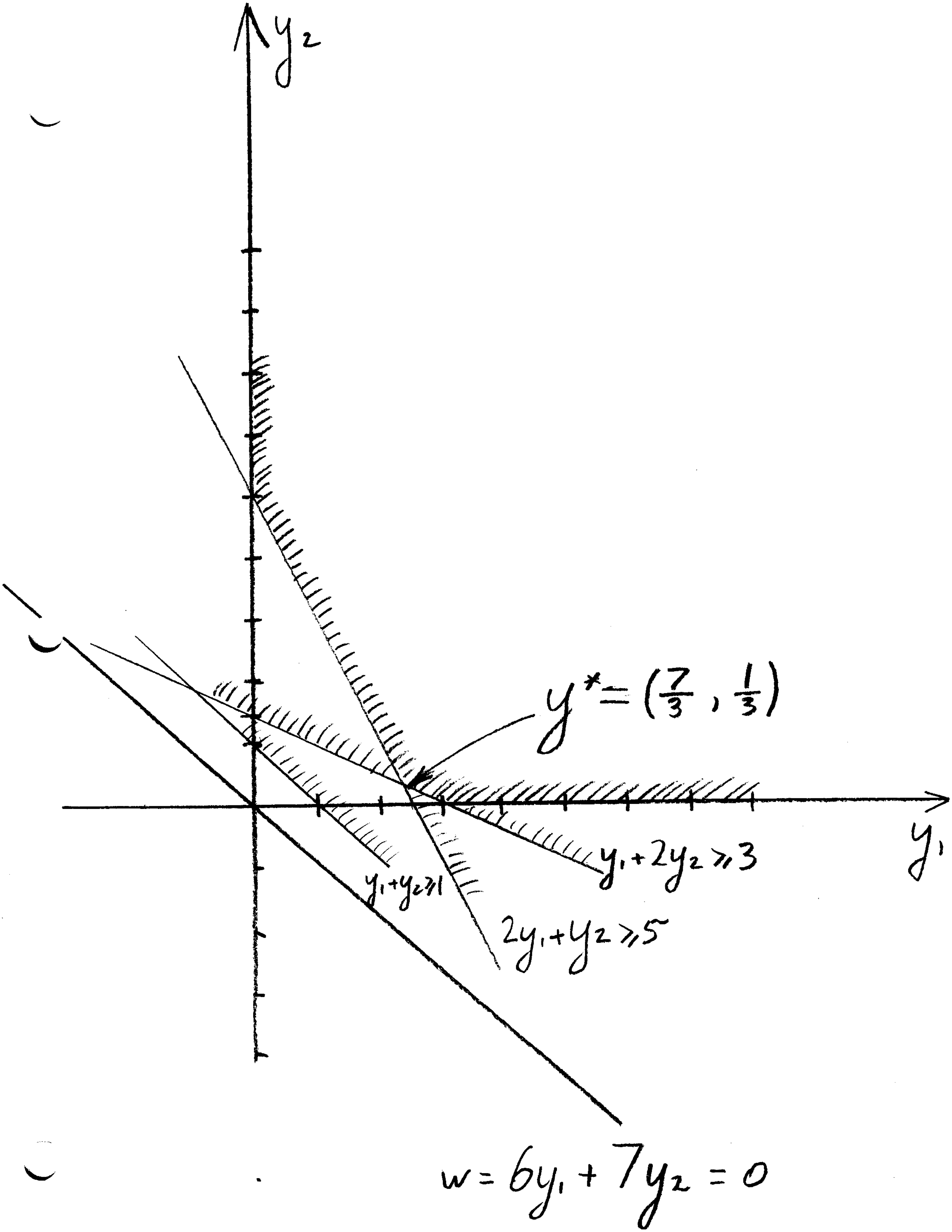
$$y_1 + y_2 \ge 1$$

$$y_1 \ge 0, y_2 \ge 0$$

$$2y_1 + y_2 = 5$$

$$y_1 + 2y_2 = 3$$

$$y_1^* = \tfrac{7}{3}, \; y_2^* = \tfrac{1}{3}$$


y_2
y_1
$y^* = (\frac{7}{3}, \frac{1}{3})$
$y_1 + 2y_2 \geq 3$
$y_1 + y_2 \geq 1$
$2y_1 + y_2 \geq 5$
$w = 6y_1 + 7y_2 = 0$

$$y_1^* + y_2^* = \frac{8}{3} > 1 \Rightarrow x_3^* = 0$$

$$y_1^* > 0 \Rightarrow \quad 2x_1 + x_2 = 6$$

$$y_2^* > 0 \Rightarrow \quad x_1 + 2x_2 = 7$$

$$x_1^* = \frac{5}{3}, x_2^* = \frac{8}{3}$$

$$w^* = 6.\frac{7}{3} + 7.\frac{1}{3} = \frac{49}{3}$$

$$z^* = 5.\frac{5}{3} + 3.\frac{8}{3} = \frac{49}{3}$$

Example

$$z = 3x_1 + 5x_2 + 7x_3 + x_4 \rightarrow \max$$

$$\begin{aligned}
y_1 \qquad & x_1 + 2x_2 + 3x_3 + 5x_4 \le 10 \\
y_2 \qquad & 2x_1 + 4x_2 + x_3 + x_4 \le 15 \\
& x_i \ge 0
\end{aligned}$$

$$w = 10y_1 + 15y_2$$

$$\begin{aligned}
& y_1 + 2y_2 \ge 3 \\
& 2y_1 + 4y_2 \ge 5 \\
& 3y_1 + y_2 \ge 7 \\
& 5y_1 + y_2 \ge 1 \\
& y_1 \ge 0,\ y_2 \ge 0
\end{aligned}$$

$$2y_1 + y_2 = 1 \quad \Big| \quad \Big| \quad 2y_1 + y_2 = 1$$

$$y_1 + 3y_2 = 1 \quad \Big|\, 2 \,\Big| \quad 2y_1 + 6y_2 = 2$$

$$5y_2 = 1, \quad y_2^* = \frac{1}{5}$$

$$2y_1 + \frac{1}{5} = 1 \qquad y_1^* = \frac{2}{5}$$

$$3 \cdot \frac{1}{5} + 2\,\frac{2}{5} > 1 \longrightarrow x_3^* = 0$$

$$y_1^* > 0 \Rightarrow 2x_1 + x_2 = 12 \quad \Big| \quad \Big| \quad 2x_1 + x_2 = 12$$

$$y_2^* > 0 \Rightarrow x_1 + 3x_2 = 18 \quad \Big|\, 2 \,\Big| \quad 2x_1 + 6x_2 = 36$$

$$5x_2 = 24$$

$$x_2^* = \frac{24}{5}$$

$$2x_1 + \frac{24}{5} = 12$$

$$2x_1 = 12 - 4\frac{4}{5} = 7\frac{1}{5}$$

$$x_1^* = \frac{18}{5}$$

Primal $z = x_1 + x_2 + x_3 \to max$

$$2x_1 + x_2 + 3x_3 \le 12$$

$$x_1 + 3x_2 + 2x_3 \le 18$$

$$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0$$

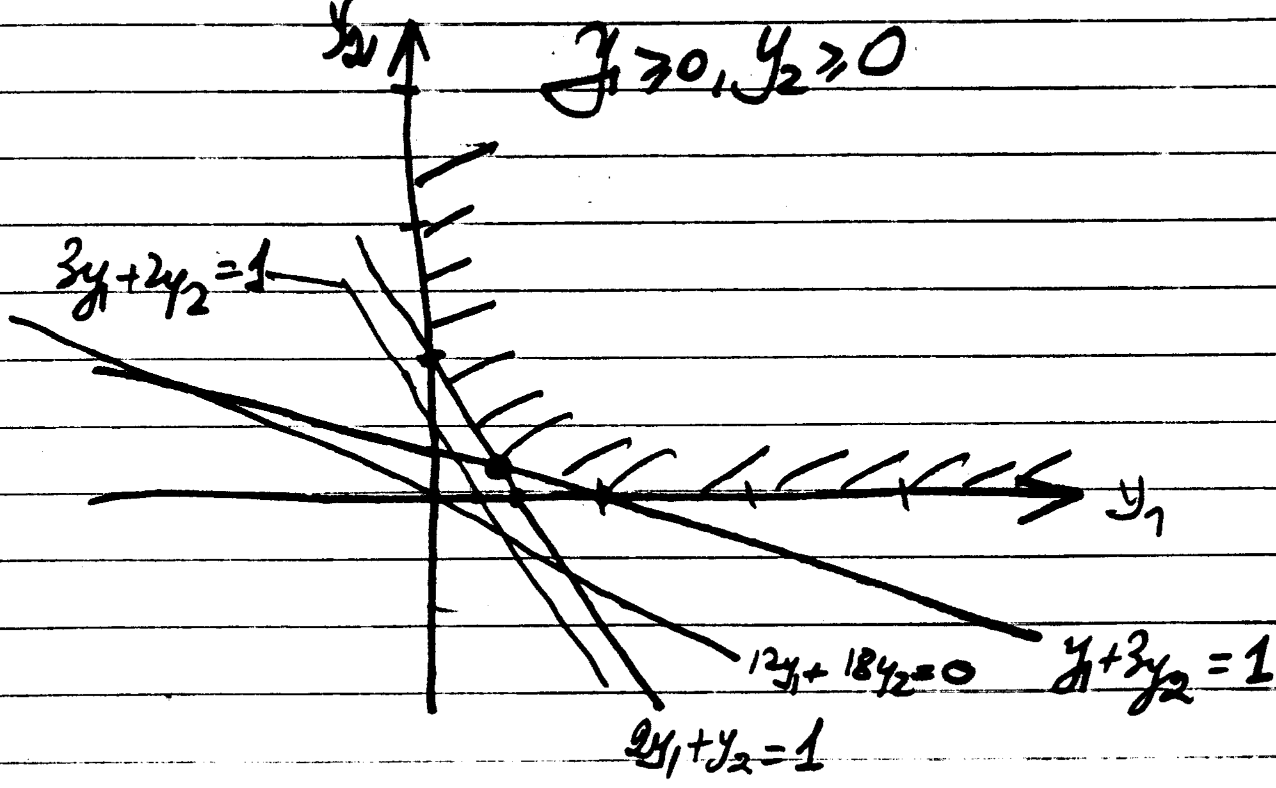
Dual $w = 12y_1 + 18y_2 \to min$

$$2y_1 + y_2 \ge 1$$

$$y_1 + 3y_2 \ge 1$$

$$3y_1 + 2y_2 \ge 1$$

$$y_1 \ge 0,\ y_2 \ge 0$$



Dual pair (x^*, y^*)-saddle point of the Lagrangean

$$c^T x \rightarrow max \qquad b^T y \rightarrow min$$

$$Ax \leq b \qquad A^T y \leq c$$

$$x \geq 0 \qquad y \geq 0$$

$$L(x, y) = (c, x) - (y, Ax - b)$$

(x^*, y^*)-saddle point

$$(c, x) - (y^*, Ax - b) \leq (c, x^*) - (y^*, Ax^* - b)$$
$$\leq (c, x^*) - (y, Ax^* - b) \quad \forall x \in R^n_+, y \in R^m_+$$

Primal-Dual System

$$Ax \leq b \qquad A^T y \geq c$$

$$x \geq 0 \qquad y \geq 0$$

$$+$$

$$c^T x \geq b^T y$$

$$\Downarrow$$

x-primal feasible y-dual feasible

$$c^T x = b^T y$$

$$x = x^*, y = y^*$$

Rules

1. max $\Rightarrow$ min
2. $c \rightarrow$ r.h.s., $b \rightarrow$ o.f.
3. $\leq \Rightarrow \geq 0$
4. $A \rightarrow A^T$
5. The num. of primal var. = num. of dual constr.

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$$z = (c,x) \rightarrow \max \qquad w = (b,y) \rightarrow \min$$

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$$x \geq 0 \qquad y \geq 0$$