Standard form of LP

$$\begin{gathered}\max z = c^T x \\ Ax = b \\ x \geq 0\end{gathered}$$

If we have a min. problem

$$\begin{gathered}\min z = c^T x \\ Ax = b \\ x \geq 0\end{gathered}$$

$$\Updownarrow$$

$$\begin{gathered}\max z = -c^T x \\ Ax = b \\ x \geq 0\end{gathered}$$

Converting LP into standard form

1) $$\begin{aligned} z = c^T x &\rightarrow \text{max} \\ Ax &\le b \\ x &\ge 0 \end{aligned}$$

$$\begin{aligned} z = c^T x + 0^T s &\rightarrow \text{max} \\ Ax + s &= b \\ x &\ge 0 \end{aligned}$$

2) $$\begin{aligned} z = c^T x &\rightarrow \text{min} \\ Ax &\ge b \\ x &\ge 0 \end{aligned}$$

$$e = Ax - b \geq 0$$

$$or$$

$$Ax - e = b$$

$$z = c^T x + 0^T e$$

$$Ax - e = b$$

$$x \geq 0,\ e \geq 0$$

3)

$$z = c^T x \to \max \qquad z = c^T x + 0^T s + 0^T e$$

$$A_1 x \leq b_1 \qquad A_1 x + s = b_1$$

$$A_2 x \geq b_2 \qquad A_2 x - e = b_2$$

$$x \geq 0 \qquad x \geq 0,\ s \geq 0,\ e \geq 0$$

$$
\begin{aligned}
z = c^T x &\to \max \\
Ax &\le b \\
x &\ge 0
\end{aligned}
$$

$$
\begin{aligned}
z = c^T x + 0^T s &\to \max \\
Ax + s &= b \\
x \ge 0,\ s &\ge 0
\end{aligned}
$$

$$
\begin{array}{cccccc}
x_1 & x_j & x_n & s_1 & s_m & \\
a_{11} & a_{1j} & a_{1n} & 1 & 0 & b_1 \\
a_{i1} & a_{ij} & a_{in} & & & b_i \\
a_{m1} & a_{mj} & a_{mn} & o & 1 & b_m \\
-c_1 & -c_j & -c_n & 0 & 0 &
\end{array}
$$

Multiple Solution

$$\begin{array}{lcccc}
x_j & x_1 & x_2 & x_m & 1 \\
a_{1j} & 1 & 0 & 0 & b_1 \\
a_{ij} & 0 & 1 & 0 & b_i \\
a_{mj} & 0 & 0 & 1 & b_m \\
0 & 0 & 0 & 0 & z
\end{array}$$

Degeneracy

$$\begin{array}{lcccc}
x_j & x_1 & .. & x_m & 1 \\
a_{1j} & 1 & 0 & 0 & b_1 \\
a_{ij} & 0 & 1 & 0 & 0 \\
a_{mj} & 0 & 0 & 1 & b_m \\
\Delta_j > 0 & & & & z
\end{array}$$

Bland's Rule

1. Among all nonbasic variables with negative reduced costs, choose the one with the smallest index to enter the basis.
2. When there is a tie in the minimum ratio test, choose the basic variable with the smallest index to leave the basis.

Big M method

$$c^T x \to \min$$

$$Ax = b$$

$$x \geq 0$$

$$c^T x + M^T y \to \min$$

$$Ax + y = b$$

$$x \geq 0, \quad y \geq 0$$

	c_1		c_j		c_n	M		M	
	x_1	..	x_j	..	x_n	y_1	..	y_m	1
			a_{1j}			1		0	
M			a_{ij}						b
			a_{mj}			0		1	
			Δ_j						

$$\Delta_j = M.a_{1j} + \ldots + Ma_{mj} - c_j$$

$$if \ \ \Delta_j > 0 \Rightarrow use\ the\ ratio\ test$$

$$i_0: \quad \frac{b_{i0}}{a_{i0\,j}} = \min\left\{ \frac{b_i}{a_{ij}} : a_{ij} > 0 \right\}$$

After m steps

y_1	..	y_m	x_{m+1}	..	x_n	x_1	..	x_m	1
						1		0	$b_1^{'}$
			A_{m+1}	..	A_n				
						0		1	$b_m^{'}$
			Δ_{m+1}		Δ_n				

$$c^T x \to \min$$

$$\begin{aligned}
A_1 x &\le b_1 \\
A_2 x &= b_2 \\
A_3 x &\ge b_3 \\
x &\ge 0
\end{aligned}$$

$$\begin{aligned}
&c^T x + 0s_1 + My_2 + 0a_3 + My_3 \to \min \\
&A_1 x + s_1 \qquad\qquad\qquad\qquad = b_1 \\
&A_2 x + \quad + y_2 \qquad\qquad\quad = b_2 \\
&A_3 x \qquad\qquad\quad - a_3 + y_3 = b_3 \\
&x \ge 0, s_1 \ge 0, y_2 \ge 0, a_3 \ge 0, y_3 \ge 0
\end{aligned}$$

$$x_1 + 2x_2 \to \min$$

$$x_1 + x_2 \leq 16$$

$$x_1 + 3x_2 \geq 20$$

$$2x_1 + x_2 = 10$$

$$x_1 \geq 0, x_2 \geq 0$$

$$
\begin{array}{llll}
x_1 + 2x_2 & & + Ma_1 + Ma_2 & \rightarrow \min \\
x_1 + x_2 + s_1 & & & = 16 \\
x_1 + 3x_2 & - e_1 + a_1 & & = 20 \\
2x_1 + x_2 & & + a_2 & = 10 \\
x_1 \geq 0, \ldots, \ a_2 \geq 0 & & &
\end{array}
$$

$$
\begin{array}{cccccccc}
 & x_1 & x_2 & s_1 & e_1 & a_1 & a_2 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 16 \\
M & 1 & 3 & 0 & -1 & 1 & 0 & 20 \\
M & 2 & 1 & 0 & 0 & 0 & 1 & 10 \\
 & 1 & 2 & 0 & 0 & M & M & 0
\end{array}
$$

x_1	x_2	s_1	e_1	a_1	a_2	1
1	1	1	0	0	0	16
1	3	0	-1	1	0	20
2	1	0	0	0	1	10
$3M-1$	$4M-2$	0	$-M$	0	0	30M

a_2	x_2	s_1	e_1	a_1	x_1	1	
-1	1	1	0	0	0	22	
-1	5	0	-2	1	0	30	:2
1	1	0	0	0	1	10	
$1-3M$	$5M-3$	0	$-2M$	0	0	30M+10	

$$\begin{array}{ccccccc}
a_2 & x_2 & s_1 & e_1 & a_1 & x_1 & \\
\frac{-1}{2} & \frac{1}{2} & 1 & 0 & 0 & 0 & 11 \\
\frac{-1}{2} & \enclose{circle}{\frac{5}{2}} & 0 & -1 & 1 & 0 & 15 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 1 & 5 \\
\frac{1-3M}{2} & \frac{5M-3}{2} & 0 & -M & 0 & 0 & 15M+5
\end{array}$$

$$\begin{array}{cccccc}
e_1 & s_1 & x_2 & x_1 & 1 & \\
 & 1 & 0 & 0 & \frac{55}{2} - \frac{15}{2} & \\
 & 0 & 1 & 0 & 15 & :\frac{5}{2} \\
 & 0 & 0 & 1 & \frac{25}{2} - \frac{15}{2} & \\
-\frac{3}{2} & 0 & 0 & 0 & 35 &
\end{array}$$

$x_1 = 2, \; x_2 = 6, \; s_1 = 8$

Two phase method

$-4x_1 + x_2 \quad - x_3 \rightarrow \min$

$3x_1 + x_2 + x_3 = 6$

$-x_1 + 2x_2 - x_3 = 4$

$x_i \geq 0$

First phase

$a_1 + a_2 \rightarrow \min$

$3x_1 + x_2 + x_3 + a_1 \qquad = 6$

$-x_1 + 2x_2 - x_3 \qquad + a_2 = 4$

$x_i \geq 0, a_i \geq 0$

$$\begin{array}{ccccccc}
 & 0 & 0 & 0 & 1 & 1 & \\
 & x_1 & x_2 & x_3 & a_1 & a_2 & 1 \\
1 & 3 & 1 & 1 & 1 & 0 & 6 \\
1 & -1 & \textcircled{2} & -1 & 0 & 1 & 4 \\
\Delta_j & 2 & 3 & 0 & 0 & 0 &
\end{array}$$

$$\Delta_1 = 1.3 + 1.(-1) - 0 = 2$$

$$\Delta_2 = 1.1 + 1.2 - 0 = 3$$

$$\Delta_3 = 1.1 + 1.(-1) - 0 = 0$$

$$\begin{array}{ccccccc}
x_1 & a_2 & x_3 & a_1 & x_2 & 1 & \\
7 & -1 & 3 & 1 & 0 & 8 & \\
-1 & 1 & -1 & 0 & 1 & 4 & :2 \\
7 & -3 & 3 & 0 & 0 & &
\end{array}$$

$$\begin{array}{cccccc}
x_1 & a_2 & x_3 & a_1 & x_2 & 1 \\
\enclose{circle}{\frac{7}{2}} & \frac{-1}{2} & \frac{3}{2} & 1 & 0 & 4 \\
\frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} & 0 & 1 & 2 \\
\frac{7}{2} & \frac{-3}{2} & \frac{3}{2} & 0 & 0 &
\end{array}$$

$$\begin{array}{cccccc}
a_1 & a_2 & x_3 & x_1 & x_2 & 1 \\
1 & \frac{-1}{2} & \frac{3}{2} & 1 & 0 & 4 \\
\frac{1}{2} & \frac{6}{4} & -1 & 0 & 1 & 9 \\
\frac{-7}{2} & \frac{-14}{4} & 0 & 0 & 0 &
\end{array}
\quad : \frac{7}{2}$$

$$\begin{array}{cccc}
x_3 & x_1 & x_2 & \\
\frac{3}{7} & 1 & 0 & \frac{8}{7} \\
\frac{-2}{7} & 0 & 1 & \frac{18}{7} \\
0 & 0 & 0 &
\end{array}$$

$$\begin{array}{ccccc} & x_1 & x_2 & x_3 & 1 \\ -4 & 1 & 0 & \frac{3}{7} & \frac{8}{7} \\ 1 & 0 & 1 & \frac{-2}{7} & \frac{18}{7} \\ & 0 & 0 & -1 & -2 \end{array}$$

$$\Delta_3 = -4.\tfrac{3}{7} + 1\left(\tfrac{-2}{7}\right) - (-1) = -1$$

$$-4x_1 + x_2 - 10x_3 \to \min$$

$$3x_1 + x_2 + x_3 = 6$$

$$-x_1 + 2x_2 - x_3 = 4$$

$$x_i \geq 0$$

$$\Delta_3 = -4.\tfrac{3}{7} + 1.\left(\tfrac{-2}{3}\right) - (-10) = 8$$

$$\begin{array}{cccccc} & x_1 & x_2 & x_3 & & \\ -4 & 1 & 0 & \frac{3}{7} & \frac{8}{7} & : \frac{3}{7} \\ 1 & 0 & 1 & \frac{-2}{7} & \frac{18}{7} & \\ & 0 & 0 & 8 & -2 & \end{array}$$

	x_3	x_2	x_1	
-10	1	0	$\frac{7}{3}$	$\frac{8}{3}$
1	0	1	$-\frac{2}{3}$	$\frac{10}{3}$
	0	0	$-\frac{8.7}{3}$	$\frac{-70}{3}$

$$x_1^* = 0,\ x_2^* = \frac{10}{3},\ x_3^* = \frac{8}{3}$$

$$z^* = -4.0 + \frac{10}{3} - 10.\frac{8}{3} = \frac{-70}{3}$$

x_1	x_2	x_3	x_4	x_5	x_6	x_7	1
2	5	1	d	0	-2	0	1
-2	f	0	-1	1	$-e$	0	e
1	2	0	2	0	0	1	d
a	b	0	h	0	c	0	

Maximization problem

a) Current solution is optimal

b) Current solution is unique optimal

c) Current is optimal but alternate optimal exists

d) The problem is unbounded

e) The current solution is not optimal

x_4 → in x_3 → out

f) The current solution is not optimal, when x_2 is in x_5 is out but the function value remains the same

a) $\min\{e,d\} \geq 0$, $\min\{a, b, c, h\} \geq 0$

b) $\min\{a, b, c, h\} > 0$

c) $\min\{a, b, c, h\} = 0$

d) $c < 0$

e) $\min\{a, b, c, h\} < 0$, $\min\{1/d, d/2\} = 1/d$, $d > 0$

f) $\min\{a, b, c, h\} < 0$, $\min\{e/f, d/2, 1/5\} = e/f$,
$f > 0$, $b = 0$ or $e = 0$ or both

$$\begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & 1 \\ 3 & 7 & 1 & d & 0 & -e & 0 & e \\ -2 & f & 0 & -1 & 1 & -2 & 0 & d \\ 1 & a & 0 & 2 & 0 & 0 & 1 & 1 \\ a & b & 0 & r & 0 & c & 0 & \end{array}$$

1) Current solution is optimal
2) Current solution is unique optimal
3) There is an alternative opt solution
4) The problem is unbounded
5) The current sol. is not opt

$x_4 \to$ in $\quad x_3$ − out

$$c^T x \to \max$$

$$Ax = b$$

$$x \geq 0$$

$$\begin{array}{cccc}
 & c_B & c_N & 1 \\
B^{-1} & \boxed{B} & \boxed{N} & b
\end{array}$$

$$\begin{array}{cccc}
 & c_B & c_j & \\
c_B & \boxed{I} & \boxed{B^{-1}A_j} & B^{-1}b \\
 & & \Delta_j &
\end{array}$$

$$\Delta_j = c_B B^{-1} A_j - c_j = y^T A_j - c_j \geq 0$$

$$\Delta_N = y^T N - c_N \geq 0$$

$$\Delta_B = y^T B - c_B = c_B B^{-1} B - c_B = 0$$

$$\begin{array}{c|c|c}
I & B^{-1} A_j & B^{-1} b \\
\hline
0 & \Delta_j = y^T A_j - c_j &
\end{array}$$

$$x_B = x_B^* = B^{-1} b, \qquad x_N^* = 0$$

$$\begin{array}{c|ccc|c|c}
 & & c_B & & c_j & \\ \hline
c_1 & 1 & .. & 0 & & \\
 & & & & B^{-1}A_j & B^{-1}b \\
c_m & 0 & .. & 1 & & \\ \hline
 & & & & \Delta_j &
\end{array}$$

$$\Delta_1 = c_1.1 + \ldots c_m.0 - c_1 = 0$$

— — — — — — — — — — — — — — — —

$$\Delta_m = c_1.0 + \ldots c_m.1 - c_m = 0$$

$$\Delta_j = c_B B^{-1} A_j - c_j$$

$$If \ \ \Delta_j < 0 \Rightarrow c_B B^{-1} A_j < c_j \Rightarrow$$

$$A_j \ \text{int}\,roduce \quad \text{int}\,o \ the \ basis$$

If $\Delta_j \geqslant 0$ *then* $x_B = B^{-1}b,\ x_N = 0$ *is the optimal solution*

1) *Changing* b

The change of b *has no effect on* Δ_j

For B to remain optimal basis

$$x_B = B^{-1}(b + \Delta b) \geq 0$$

$$\text{Consider } \Delta b = \begin{pmatrix} 0 \\ \cdot\cdot \\ \Delta b_i \\ \cdot\cdot \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot\cdot \\ \Delta \\ \cdot\cdot \\ 0 \end{pmatrix}$$

$$x_B = B^{-1}(b + \Delta) \geq 0$$

We have a system of m inequal. for Δ

By solving the system we find the range for Δ

For each $b_i \Rightarrow b_i + \Delta$ there is a system

$$B^{-1}(b_i + \Delta) \geq 0$$

$$\begin{array}{ll} z^*(b), & z^*(b+\Delta b) = z^*(b+\Delta b_i) \\ \| & \qquad \| \\ (c_B, B^{-1}b) & (c_B, B^{-1}(b+\Delta b_i)) \end{array}$$

$$y = c_B B^{-1}$$

Let $\Delta b = \left(\Delta b_1, 0, \ldots, 0\right)$

$$x_B = B^{-1}\left(b + \begin{pmatrix} \Delta b_1 \\ 0 \\ \ldots \\ 0 \end{pmatrix} \right) \geq 0$$

We have m inequalities for Δb_1

$\Delta b = \left(0, \ldots, \Delta b_i, \ldots 0\right)$

$$x_B = B^{-1}\left(b + \begin{pmatrix} 0 \\ \ldots \\ \Delta b_i \\ \ldots \\ 0 \end{pmatrix} \right) \geq 0$$

(

Let $\Delta b = (\Delta b_1, 0, \ldots, 0)$

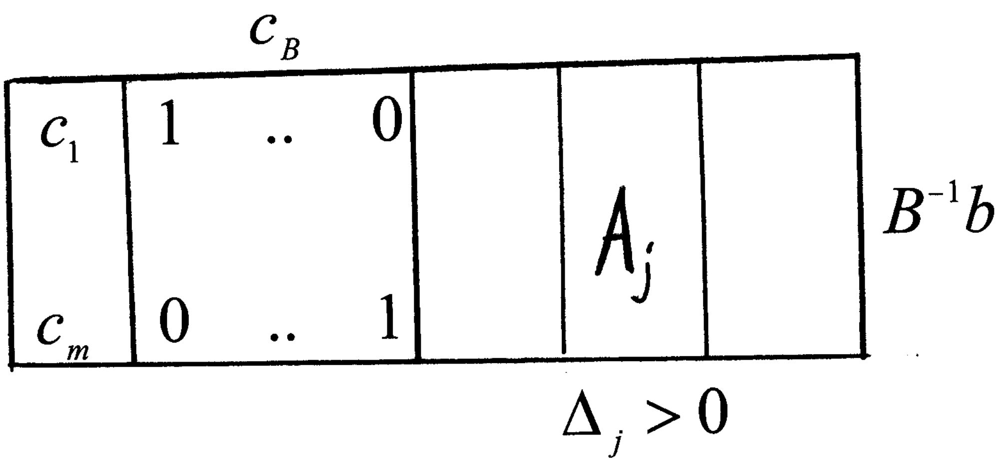
$$x_B = B^{-1}\left(b + \begin{pmatrix} \Delta b_1 \\ 0 \\ \ldots \\ 0 \end{pmatrix}\right) \geq 0$$

We have m inequalities for Δb_1

$\Delta b = (0, \ldots, \Delta b_i, \ldots 0)$

(

$$x_B = B^{-1}\left(b + \begin{pmatrix} 0 \\ \ldots \\ \Delta b_i \\ \ldots \\ 0 \end{pmatrix}\right) \geq 0$$



$$c_B B^{-1} A_j > c_j$$

$y = c_B B^{-1}$ – *shadow price*

Finding $\Delta_j = cB^{-1}A_j - c_j$ – *reduced* cost

or pricing out

Sensitivity Analysis

$$z = c^T x \rightarrow \max$$

$$Ax = b$$

$$x \geq 0$$

$$z^*(A,b,c), \quad z^*(b), \quad z^*(c)$$

A is always fixed

$$\begin{array}{ccc} c_B & c_N & \\ \cline{1-2} \multicolumn{1}{|c|}{B} & \multicolumn{1}{c|}{\quad N \quad} & b \\ \cline{1-2} \end{array}$$

$$B^{-1} \textit{ exists} \qquad B^{-1}Ax = B^{-1}b$$