LP standard form

$$Z = c^T x \to \max$$

$$Ax \leq b$$

$$x \geq 0$$

$$A = m\left(\overset{n}{A_1, \ldots, A_r, \underset{n > m}{A_{r+1}}, \ldots, A_n} \right)$$

$x = (x_1, \ldots x_r, 0, \ldots, 0)$ is a basic solution if $A_1, \ldots A_r$ are linearly independent

<u>Theorem</u> The set of extreme points are in one to one correspondence with the set of basic solutions.

Extreme point $\Rightarrow$ Basic solution

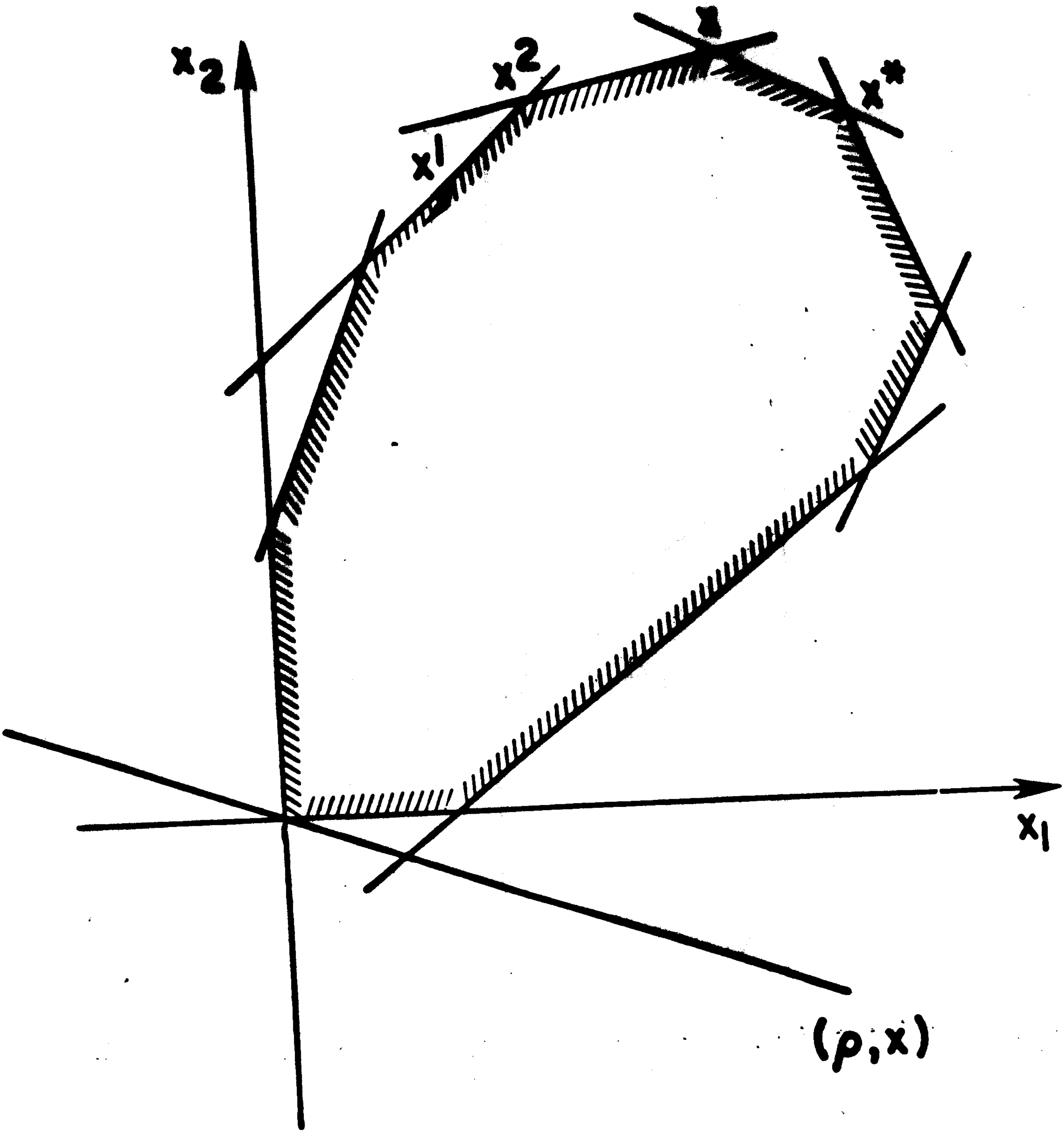
$$Let\, x^0 = \left(x_1^0, \ldots, x_r^0; 0, \ldots, 0\right) - extreme\; point$$

To prove that x^0 is a basic solution we have to prove that

$A_1, \ldots, A_r$ are linear independent

$$x_1 A_1 + \ldots + x_r A_r = 0, \quad x_1 \neq 0$$

$$x^1 = \left(x_1^0 + tx_1, \ldots, x_r^0 + tx_r; 0, \ldots, 0\right)$$
$$x^2 = \left(x_1^0 - tx_1, \ldots, x_r^0 - tx_r; 0, \ldots, 0\right)$$
$$for\; small\; t \rangle 0$$
$$x^1 \geq 0, x^2 \geq 0\; and$$


x_2
x^2
x^1
x^*
x_1
(p, x)

$$x^0 = \frac{x^1 + x^2}{2}$$

So x^0 is not an extreme point

$A_1, \ldots, A_r$ are linear independent

The second part

X^0 – basic solution x^0 – extreme point

$$x^0 = \left(x_1^0, \ldots, x_r^0; 0, \ldots, 0\right)$$

Assuming that it is not so, then

$$\forall x^1, x^2: \qquad x^1 \neq x^2$$

$$x^0 = \lambda x^1 + (1 - \lambda) x^2$$

$$x^1 = \left(x_1^1, \ldots, x_r^1; 0, \ldots, 0\right)$$

$$x^2 = \left(x_1^2, \ldots, x_r^2; 0, \ldots, 0\right)$$

$$x_1^1 A_1 + \ldots + x_r^1 A_r = b$$

$$x_1^2 A_1 + \ldots + x_r^2 A_r = b$$

$$\left(x_1^1 - x_1^2\right) A_1 + 0 \ldots + \left(x_r^1 - x_r^2\right) A_r = 0$$

$$A_1, \ldots, A_r \textit{ are linear independent}$$

$$x_1^1 = x_1^2, \ldots, x_r^1 = x_r^2$$

$$x^1 = x^2$$

$$x^0 \textit{ basic sol.} \Rightarrow x^0 - \textit{extr. point}$$

The solution of LP is a <u>VERTEX</u>

$$C^T x = C^T \left(\sum_{i=1}^{N} \lambda_i x_i \right)$$

$$C^T x_1 \geq C^T x_2 \geq \ldots \geq C^T x_N$$

$$\max C^T x = \max_{1 \leq i \leq N} C^T x_i = C^T x_1 \ !$$

$$x \in \Omega$$

Linear Programming

$$Z = C^T x \rightarrow \max$$

$$Ax \leq b$$

$$x \geq 0$$

Example

$$Z = x_1 + 2x_2 \rightarrow \max$$

$$2x_1 + x_2 \leq 4$$

$$x_1 + 3x_2 \leq 7$$

$$x_1 \geq 0, \; x_2 \geq 0$$

Moving from one vertex to another ⇔ moving from one basic solution to another

$$z = x_1 + 2x_2 \to \max.$$

$$2x_1 + x_2 \le 4$$

$$x_1 + 3x_2 \le 7$$

$$x_1 \ge 0, x_2 \ge 0$$

$$2x_1 + x_2 + s_1 \qquad = 4$$

$$x_1 + 3x_2 \qquad + s_2 = 7$$

$$x_1 \ge 0, x_2 \ge 0, s_1 \ge 0, s_2 \ge 0$$

z	x_1	x_2	s_1	s_2	1
	2	1	1	0	4
	1	3	0	1	7
	−1	−2	0	0	0

$$x^0 = (0,0,4,7)$$

$$4\begin{pmatrix}1\\0\end{pmatrix} + 7\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}4\\7\end{pmatrix}$$

$$\begin{pmatrix}2\\1\end{pmatrix} = 2\begin{pmatrix}1\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\end{pmatrix}$$

$$t\begin{pmatrix}2\\1\end{pmatrix} - 2t\begin{pmatrix}1\\0\end{pmatrix} - t\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$t\begin{pmatrix}2\\1\end{pmatrix}+(4-2t)\begin{pmatrix}1\\0\end{pmatrix}+(7-t)\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}4\\7\end{pmatrix}$$

$$t_1=\frac{4}{2}=2, t_2=\frac{7}{1}=7$$

$$t=2=\min\left\{\frac{4}{2},\frac{7}{1}\right\}$$

$$2\begin{pmatrix}2\\1\end{pmatrix}+0\begin{pmatrix}1\\0\end{pmatrix}+5\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}4\\7\end{pmatrix}$$

$$x^1=(2,0,0,5)$$

x_1	x_2	s_1	s_2	1
2	1	1	0	4
1	3	0	1	7
−1	−2	0	0	0

s_1	x_2	x_1	s_2	1	
1	1	1	0	4	:2
−1	5	0	1	10	
1	−3	0	0	4	

s_1	x_2	x_1	s_2	1
$\frac{1}{2}$	$\frac{1}{2}$	1	0	2
$-\frac{1}{2}$	$\frac{5}{2}$	0	1	5
$\frac{1}{2}$	$-\frac{3}{2}$	0	0	2

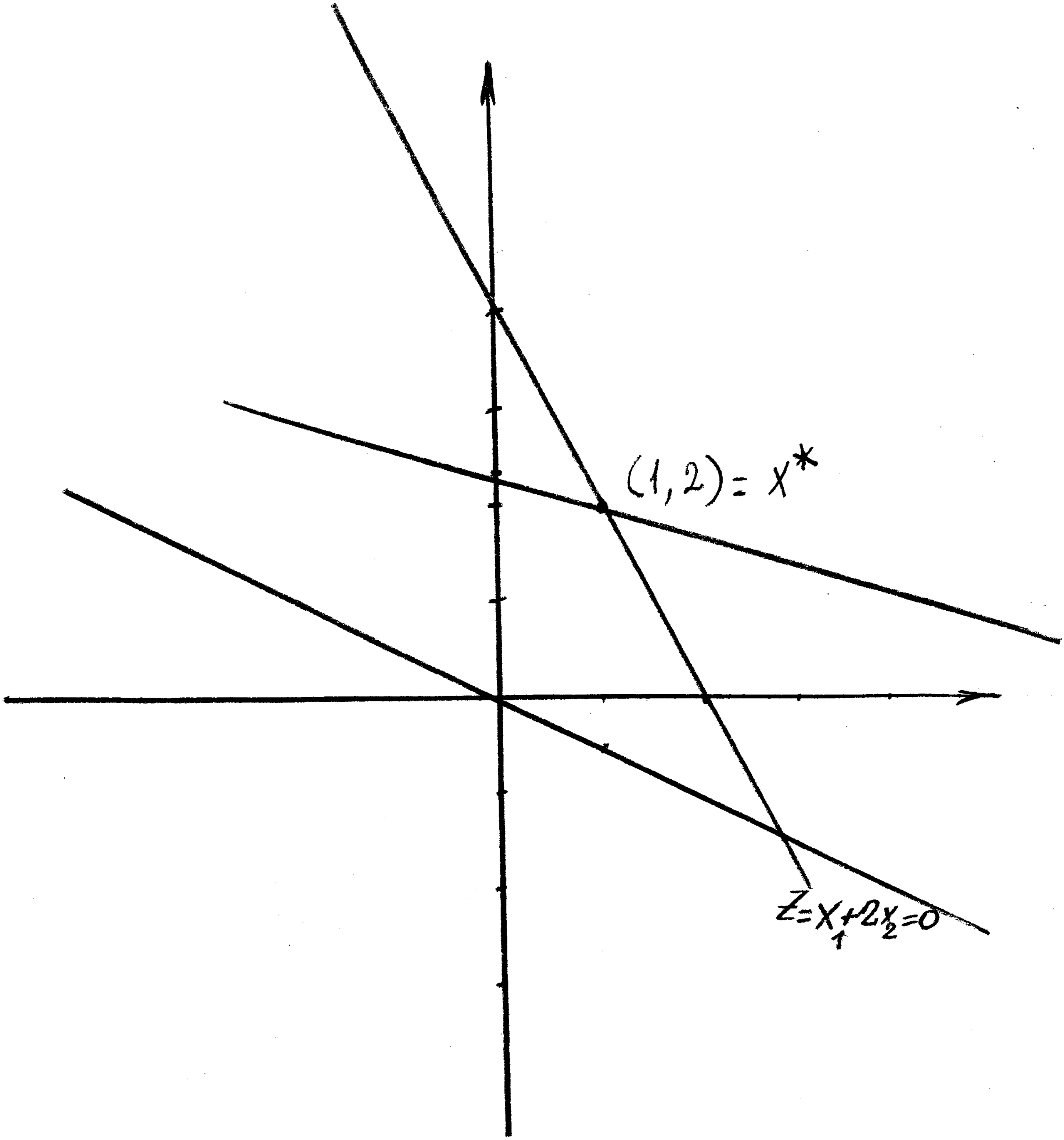
s_1	s_2	x_1	x_2	1
$3/2$	$-1/2$	1	0	$5/2$
$-1/2$	1	0	1	5
$1/2$	$3/2$	0	0	$25/2$

$:5/2$

s_1	s_2	x_1	x_2	
$3/5$	$-1/5$	1	0	1
$-1/5$	$2/5$	0	1	2
$1/5$	$3/5$	0	0	5

$$x_1^* = 1,\ x_2^* = 2$$

$$z^* = 1 + 2.2 = 5$$



$(1,2) = x^*$
$z = x_1 + 2x_2 = 0$

Dacota Co.

	Desk	Table	Chair	
Lumb	8	6	1	48
Finish.h.	4	2	1.5	20
Carp.h.	2	1.5	0.5	8
	$60	$30	$20	

$$z = 60x_1 + 30x_2 + 20x_3 \rightarrow \max$$

$$8x_1 + 6x_2 + x_3 \leq 48$$

$$4x_1 + 2x_2 + 1.5x_3 \leq 20$$

$$2x_1 + 1.5x_2 + 0.5x_3 \leq 8$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$$

x_1	x_2	x_3	s_1	s_2	s_3	1
8	6	1	1	0	0	48
4	2	1.5	0	1	0	20
(2)	1.5	0.5	0	0	1	8
−60	−30	−20	0	0	0	0

s_3	x_2	x_3	s_1	s_2	x_1		
−8	0	−2				32	
−4	−2	1				8	:2
1	1.5	0.5				8	
60	30	−10				480	

s_1	x_2	x_3	s_1	s_2	x_1	
–4	0	–1				16
–2	–1	0.5				4
0.5	0.75	0.25				4
30	15	–5				240

s_3	x_2	s_2	s_1	x_3	x_1	
						12
						4 :½
						1
⟩0	⟩0	⟩0				140

s_1	x_3	x_1	
1	0	0	24
0	1	0	8
0	0	1	2
0	0	0	280

$$x_1^* = 2,\ x_2^* = 0,\ x_3^* = 8$$

$x = 0, \; s = b$ is the first basic solution

If $\forall \; - c_j \geq 0$ then $x = 0 = x^*$ - solution because $c_j \leq 0$

If $\exists \; c_j < 0$ then find $a_{ij} > 0$

$$i_o: \min\left\{ \frac{b_i}{a_{ij}} > 0 \right\} = \frac{b_{io}}{a_{ioj}}$$

a_{ioj} is the pivot

If $\forall \; a_{ij} \leq 0$ then $\max z = \infty$

$$4x_1 + x_2 \to \max$$

$2x_1 + 3x_2 \le 4$	$2x_1 + 3x_2 + s_1 = 4$
$x_1 + x_2 \le 1$	$x_1 + x_2 + s_2 = 1$
$4x_1 + x_2 \le 2$	$4x_1 + x_2 + s_3 = 2$
$x_1 \ge 0,\ x_2 \ge 0$	$x_1 \ge 0, x_2 \ge 0, s_1 \ge 0, s_2 \ge 0, s_3 \ge 0$

	x_1	x_2	s_1	s_2	s_3	1
0	2	3	1	0	0	4
0	1	1	0	1	0	1
0	4	1	0	0	1	2
	−4	−1	0	0	0	0

$$\Delta_j = c_B^T A_j - c_j < 0$$

$$\Downarrow$$

$$c_B^T A_j < c_j$$

If $c_B^T A_j - c_j \ge 0$

then $x_B - opt.\ soln.$

	x_1	s_2	s_1	x_2	s_3	1
0	−1	−3	1	0	0	1
1	1	1	0	1	0	1
0	3	−1	0	0	1	1
	−3	1	0	0	0	1

	s_3	s_2	s_1	x_2	x_1	1		
0	1	−10	1	0	0	4		$\frac{4}{3}$
1	−1	4	0	1	0	2	:3	$\frac{2}{3}$
4	1	−1	0	0	1	1		$\frac{1}{3}$
	3	0	0	0	0	6		$\frac{6}{3}$

$$x_1^* = \frac{1}{3}, \; x_2^* = \frac{2}{3}, \quad z^* = 4.\frac{1}{3} + \frac{2}{3} = 2$$