in 1797 Lagrange published in Paris his treatise on the Theory of Analytic Functions. In this fundamental book, he clearly describes his method for the solution of constrained optimization problems:

"...when a function of several variables has to be minimized or maximized, and there are one or more equations in the variables, it will be sufficient to add the given function and the functions that must equal zero, each one multiplied by an indeterminate quantity, and then to look for the maximum or the minimum as if the variables were independent; the equations that will be found, together with the given equations, will serve for determining all the unknowns."

Constraints optimization : Equality Constraints

$$f(x) \to \min$$
$$g(x) = 0$$

Lagrangian and Lagrange Multipliers

$$L(x, \lambda) = f(x) - \lambda g(x)$$

$$\nabla_x L(x, \lambda) = \nabla f(x) - \lambda \nabla g(x) = 0$$

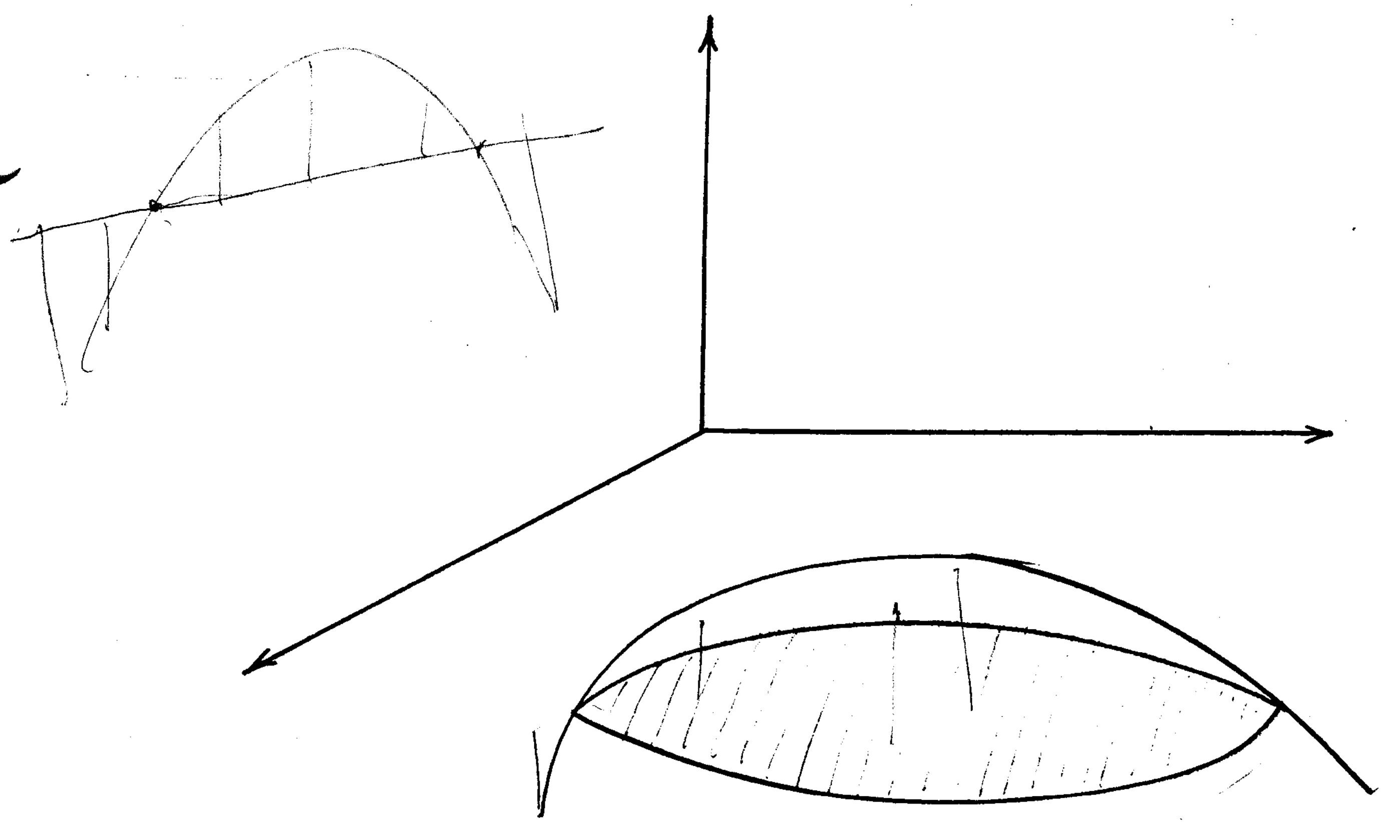
$$\nabla_\lambda L(x, \lambda) = g(x) = 0$$

Constrained Optimization: Inequality Constraints

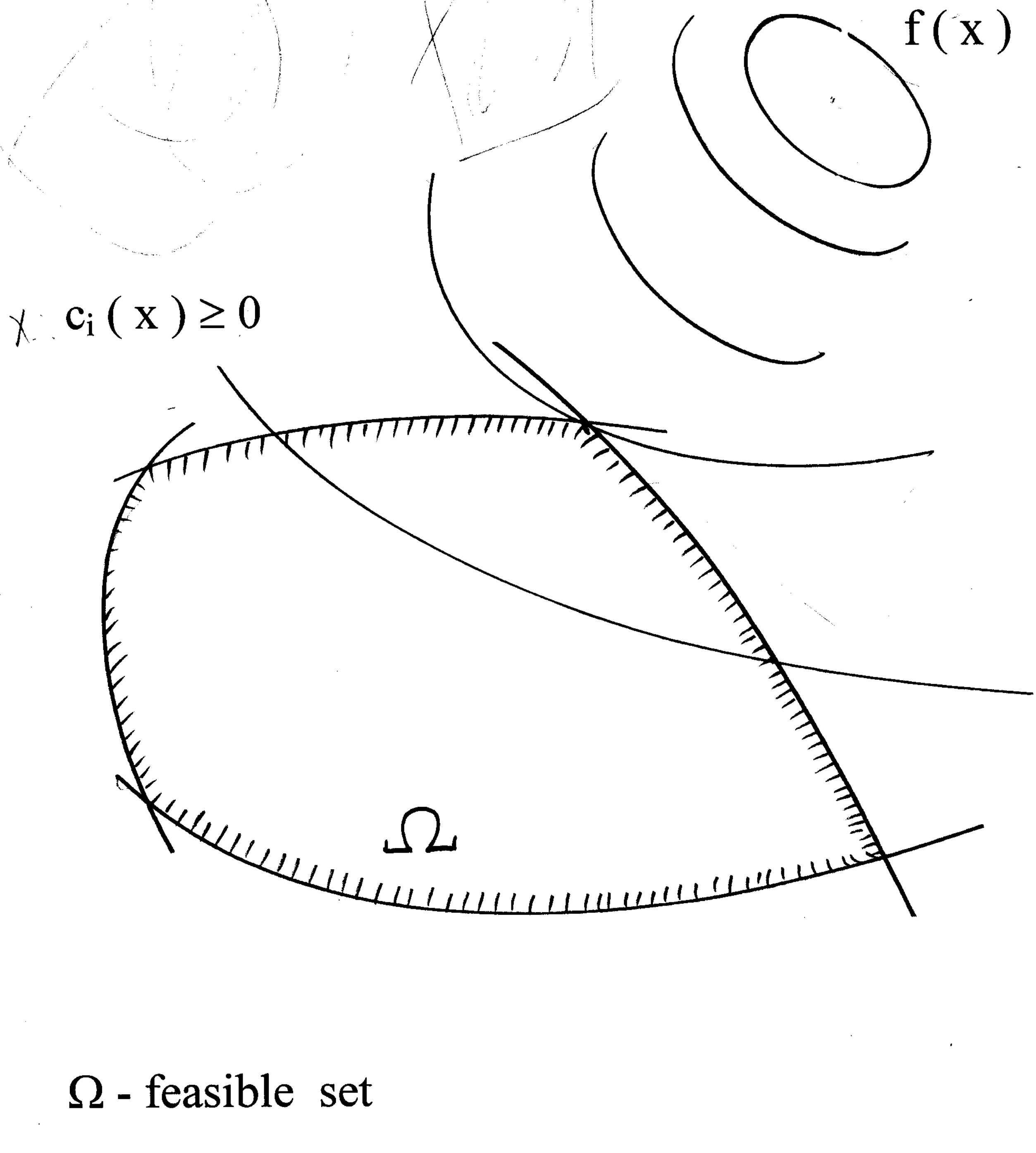
$$f(x) \to \min$$

$$c_i(x) \geq 0, \quad i = 1, \dots, m$$

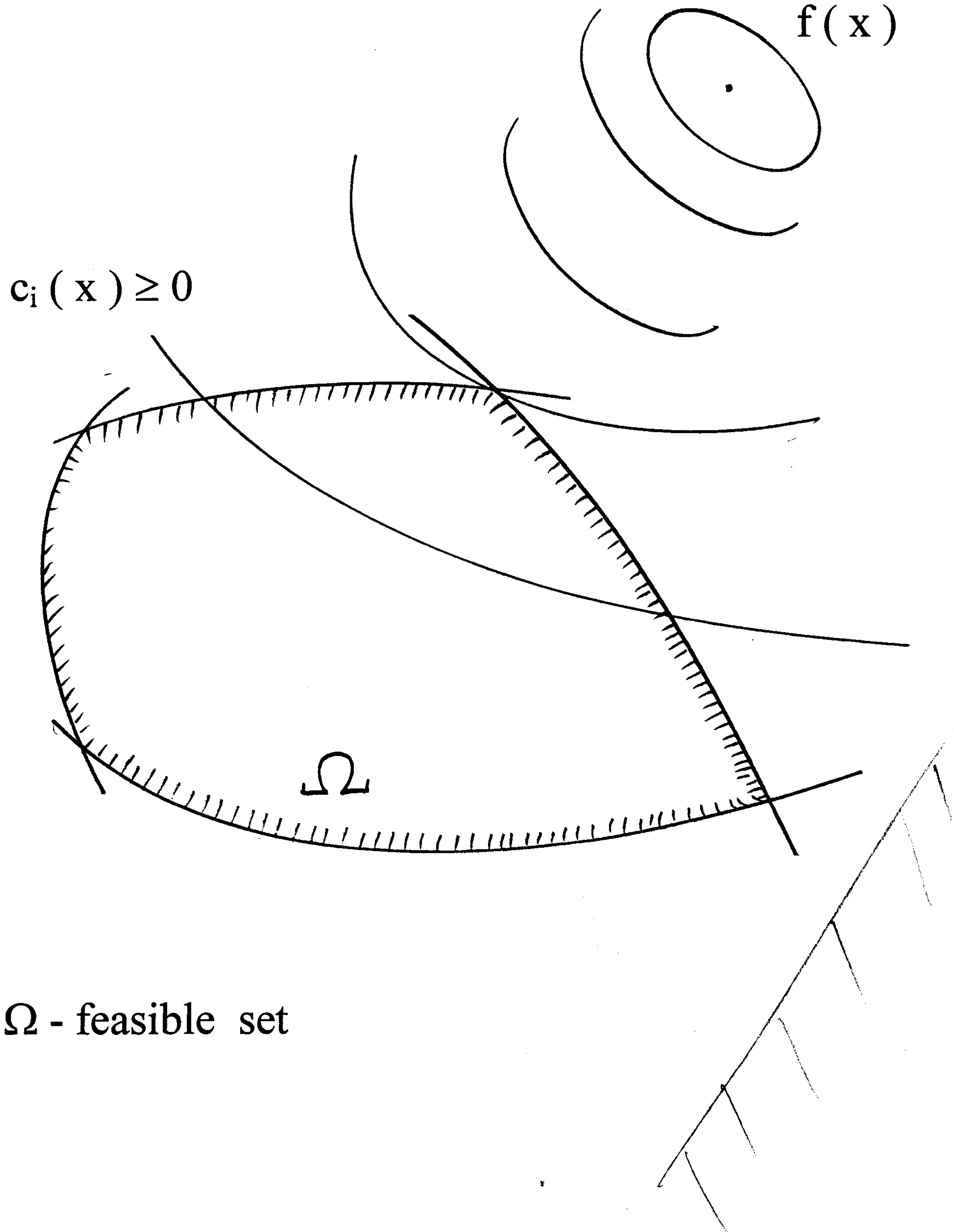
$c_i(x)$ - concave



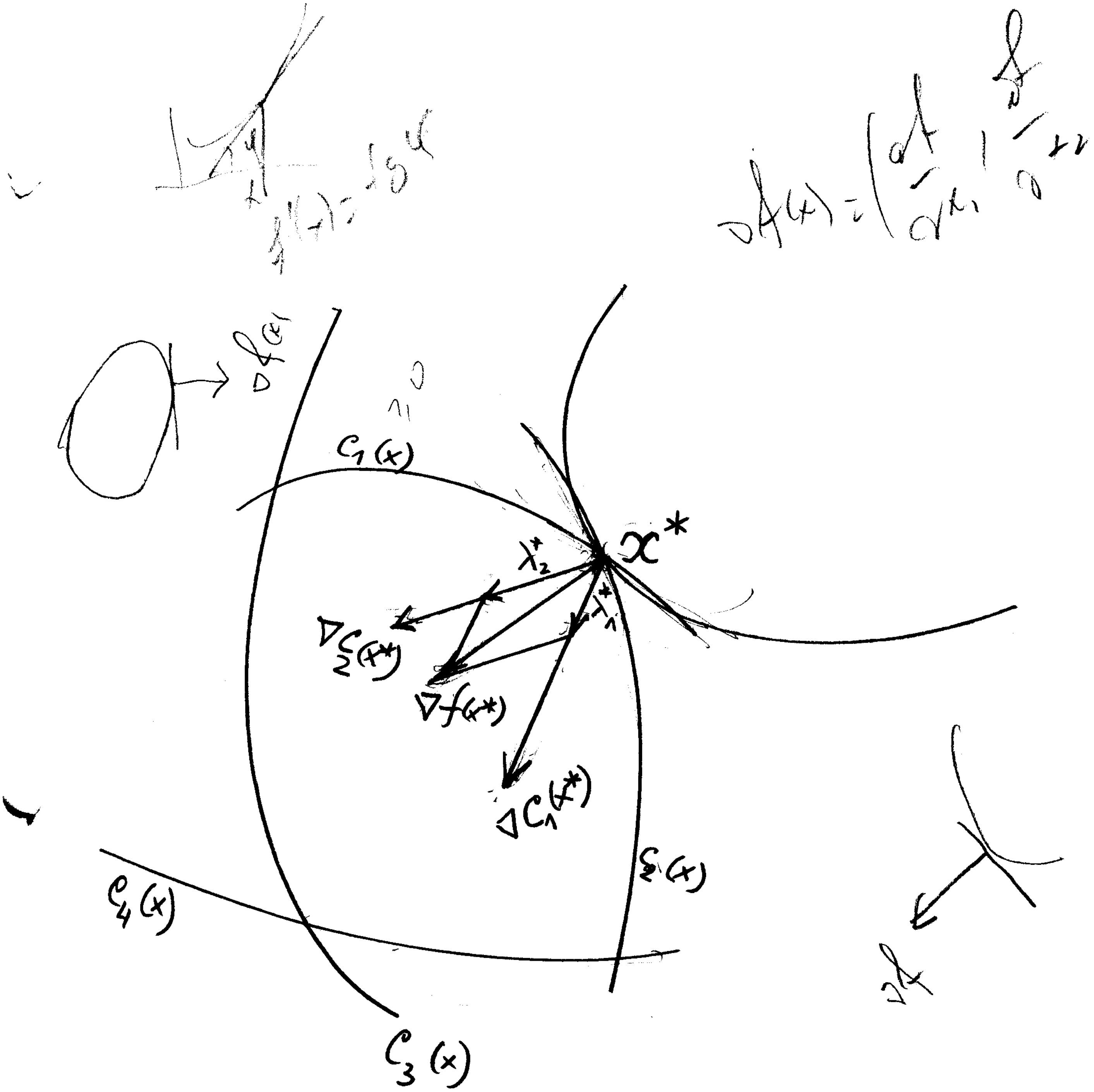
$x: \; c(x) \geq 0$



Ω - feasible set



Ω - feasible set



$$\nabla f\left(x^*\right) = \lambda_1^* \; \nabla c_1\left(x^*\right) + \lambda_2^* \; \nabla c_2\left(x^*\right)$$

$$\textit{Let} \quad \lambda_3^* = 0, \;\; \lambda_4^* = 0$$

$$\nabla f\left(x^*\right) = \sum_{i=1}^{4} \lambda_i^* \nabla c_i\left(x^*\right)$$

$$\lambda_i^* \geq 0, \quad c_i\left(x^*\right) \geq 0$$

$$\lambda_i^* c_i\left(x^*\right) = 0 , \qquad i = 1, \ldots, m$$

$$\nabla f\left(x^*\right) - \sum \lambda_i^* \nabla c_i\left(x^*\right) = 0$$

or

$$L(x, \lambda) = f(x) - \sum \lambda_i c_i(x)$$

$$\nabla_x L\left(x^*, \lambda^*\right) = \nabla f\left(x^*\right) - \sum \lambda_i^* \nabla c_i\left(x^*\right) = 0$$

$$\nabla_\lambda L\left(x^*, \lambda^*\right) = -c\left(x^*\right) \leq 0$$

$$\lambda_i^* \cdot c_i\left(x^*\right) = 0$$

Optimality Condition

Karush – Kuhn – Tucker's condition

$$L(x, \lambda) = f(x) - \sum \lambda_i c_i(x)$$

$$\begin{cases} \nabla_x \; L(x^*, \lambda^*) = 0 \\ \nabla_\lambda \; L(x^*, \lambda^*) = -c(x^*) \leq 0 \\ \lambda^{*T} c(x^*) = 0 \end{cases}$$

$$f(x) \to \min$$

$$c_i(x) \geq 0, \qquad i = 1, \ldots, m$$

$$x_j \geq 0, \qquad j = 1, \ldots, n$$

$$L(x, \lambda) = f(x) - \sum \lambda_i c_i(x)$$

$$\frac{\partial L(x, \lambda)}{\partial x_j} = \frac{\partial f(x)}{\partial x_j} - \sum \lambda_i \frac{\partial c_i(x)}{\partial x_j} \geq 0, \quad j = 1, \ldots, n$$

$$x_j \cdot \frac{\partial L}{\partial x_j} = 0$$

$$\frac{\partial L(x, \lambda)}{\partial \lambda_i} = -c_i(x) \leq 0, \qquad i = 1, \ldots, m$$

$$\lambda_i \cdot \frac{\partial L}{\partial \lambda_i} = 0$$

$$f(x_1, x_2) = e^{x_1} - \ln(x_2 + 3)$$

$$\begin{aligned} 2x_1 + 5x_2 &\le 10 \\ 3x_1 + x_2 &\le 3 \\ x_1 \ge 0,\ x_2 &\ge 0 \end{aligned}$$

$$L(x, \lambda) = e^{x_1} - \ln(x_2 + 3) - \lambda_1(10 - 2x_1 - 5x_2) - \lambda_2(3 - 3x_1 - x_2)$$

e^{x_1} – *convex because*

$$\left(e^{x_1}\right)' = e^{x_1}, \quad \left(e^{x_1}\right)'' = e^{x_1} > 0$$

$$\left(\ln(x_2 + 3)\right)' = \frac{1}{x_2 + 3}$$

$$\left(\ln(x_2 + 3)\right)'' = -\frac{1}{(x_2 + 3)^2} < 0$$

$$\left(-\ln(x_2+3)\right)'' > 0$$

$$f(x_1,\ x_2) - \text{convex}$$

$$e^{x_1} - \text{monoton. increasing}$$

$$-\ln(x_2+3) - \text{monoton. decreasing}$$

$$x_1 = 0, \quad x_2 = 2, \quad \lambda_1 = ? \quad \lambda_2 = 0$$

$$\nabla_{x_1} L = e^{x_1} + 2\lambda_1 + 3\lambda_2 \geq 0$$

$$\nabla_{x_2} L = -\frac{1}{x_2+3} + 5\lambda_1 + \lambda_2 = 0$$

$$1 + 2\lambda_1 + 3\lambda_2 \geq 0$$

$$-\frac{1}{5} + 5\lambda_1 + \lambda_2 = 0$$

$$\lambda_1 = \frac{1}{25}$$