NonLinear Optimization

K units of capital, L units of labor, KL units of goods

Capital can be purchased $4k/unit
Labor $1k/unit

Total of $8k is available

How can the firm maximize the output?

$$z = KL \rightarrow \max$$

$$4K + L \leq 8$$

$$K \geq 0, \; L \geq 0$$

Consider the LP

$$(c, x) \to \max$$
$$Ax \le b$$
$$x \ge 0$$

c is independent on x

More realistic approach c(x)

Then

$$z = (c(x), x) \to \max$$
$$Ax \le b$$
$$x \ge 0$$

Market Problem

$$p = (p_1, \ldots, p_n)$$

1) $p \in R^n_{++}$ is a positive vector of products on the market

2) m consumers:each consumer has his own vector of preferences $a_i^T = (a_{i1}, \ldots, a_{in}), 0 \leq a_{ij} \leq 1$ and a budget $b_i > 0$

$$b^T = (b_1, \ldots, b_m) \in R^m_{++}$$

Matrix of Preferences

$$A = \begin{array}{|ccccc|} \hline a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \dots & a_{mj} & \dots & a_{mn} \\ \hline \end{array}$$

Each row and each column has at least one

$$a_{ij} > 0 \qquad (*)$$

If there is a row with $a_{ij} = 0, \forall$ j then the consumer i is not interested in the market

If there is a column j with $a_{ij} = 0, \forall$ i then nobody is interested in the product j

3) For consumer i we introduce the consumptions vector
$$x_i^T = (x_{i1}, \ldots, x_{in}) \in R^n_+.$$

The components x_{ij} show how much of product j consumer i is going to buy.

4) With each consumer we associate utility function

$$u_i(x_i) = (a_i, x_i) = \sum_{j=1}^{n} a_{ij}x_{ij}$$

5) Each consumer will maximize the utility function within the budget available.
If $\pi = (\pi_1, \ldots, \pi_n)$is the vector of prices for the products then each consumer will solve the following LP problem

$$x_i^*(\pi) = argmax\{u_i(x_i)/(\pi, x_i) \leq b_i, x_i \geq 0\}$$

The total demand

$$x^*(\pi) = \sum_{i=1}^{m} x_i^*(\pi)$$

while the supply is p

Question: Is there such $\pi^* = (\pi_1^*, \ldots, \pi_n^*)$ that

$$x^*(\pi^*) = p$$

Equilibrium

D.GALE "The theory of Linear Economic Models" 1960

YES !!

For any budget vector $b \in R^m_{++}$, any supply vector $p \in R^n_{++}$ and any matrix of preferences A with property $(*)$ the equilibrium exists

Consider the following convex programming problem

$$f(x) = \sum_{i=1}^{m} b_i ln(a_i, x_i) \rightarrow max \qquad (1)$$

$$\sum_{i=1}^{m} x_i \leq p$$

and

$$x_i \in R_+^n$$

or

$$f(x) = \sum_{i=1}^{m} b_i ln(\sum_{j=1}^{m} a_{ij} x_{ij}) \rightarrow max$$

$$\sum_{i=1}^{m} x_{ij} \leq p_j$$

$$x_{ij} \geq 0, i = 1, \ldots, m; j = 1, \ldots, n$$

TARTALIA PROBLEM (1500 – 1557)

One has to divide 8 in two parts, that the sum of their product and their difference will be max.

$$x + y = 8, \quad x > 0, \quad y > 0$$

$$z = xy + (x - y) \rightarrow \max$$

$$y = 8 - x$$

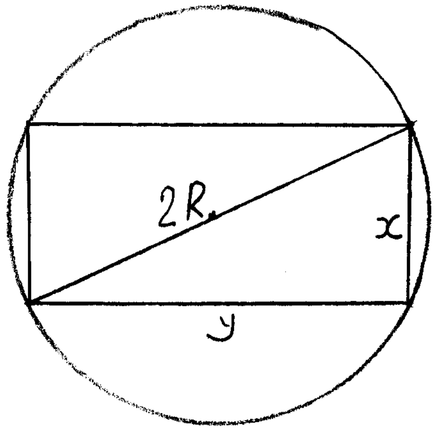
$$z = x (8 - x) + (2x - 8) \rightarrow \max$$

$$x \geq 0$$

$$z_x' = 10 - 2x = 0$$

$$x = 5, \quad y = 3$$

Inscribe rectangle with max area in a circle with a given radius R



$$z = xy \to \max$$

$$y = \sqrt{4R^2 - x^2}$$

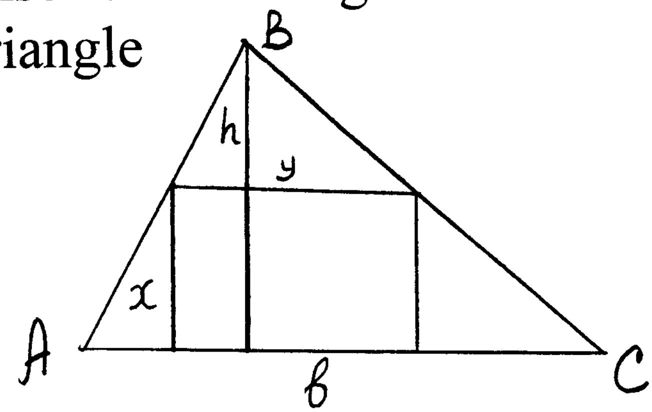
$$z = x\sqrt{4R^2 - x^{2'}} \to \max$$

$$z' = \sqrt{4R^2 - x^2} + x\frac{1}{2}\frac{-2x}{\sqrt{4R^2 - x^{2'}}}$$

$$= \frac{4R^2 - 2x^2}{\sqrt{4R^2 - x^2}} = 0$$

$$x = R\sqrt{2}, \qquad y = R\sqrt{2}$$

Inscribed rectangle with max area in a given triangle



$$z = xy$$

$$\frac{h}{H} = \frac{y}{b}, \qquad h = H - x$$

$$\frac{H-x}{H} = \frac{y}{b}, \qquad y = \frac{b}{H}(H-x)$$

$$z = \frac{b}{H} \bullet x(H-x) \to \max$$

$$x \geq 0$$

$$[x(H-x)]'_x = H - 2x = 0$$

$$x = \frac{H}{2}$$

$$y = \frac{b}{H}\left(H - \frac{H}{2}\right) = \frac{b}{2} \qquad z^* = \frac{Hb}{4}$$

$$\max \quad x_1 \dots x_n$$

$$x_1 + \dots + x_n = 1$$
$$x_i \geq 0$$

$$\sqrt[n]{x_1 \dots x_n} \leq \frac{x_1 + \dots + x_n}{n}$$

$$x_1 = \dots = x_n \qquad x_i = \frac{1}{n}$$

$$\max \; x_1 \dots . x_n = n^{-n}$$

Convex and Concave Functions

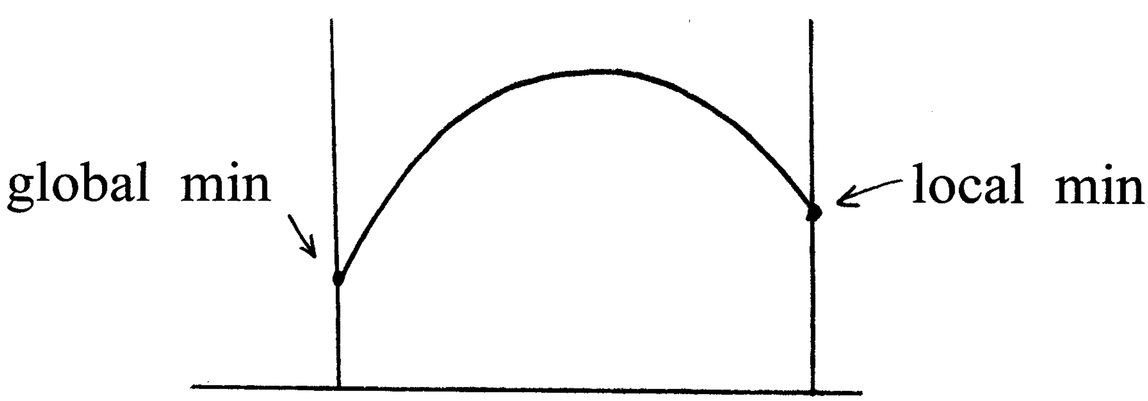
Convex Concave

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2)$$

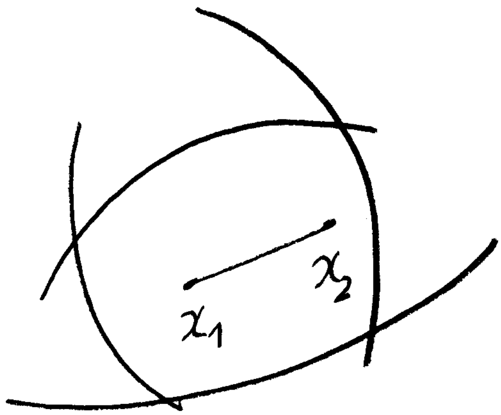
Convex

$\ge$ Concave

local min convex f(x) = global min f(x)



Convex Set



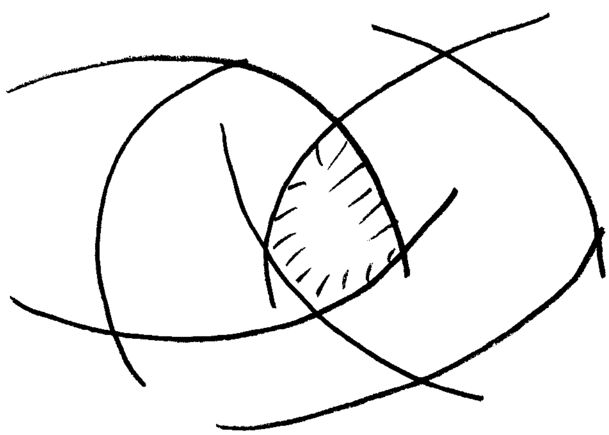
Ω *is convex if*

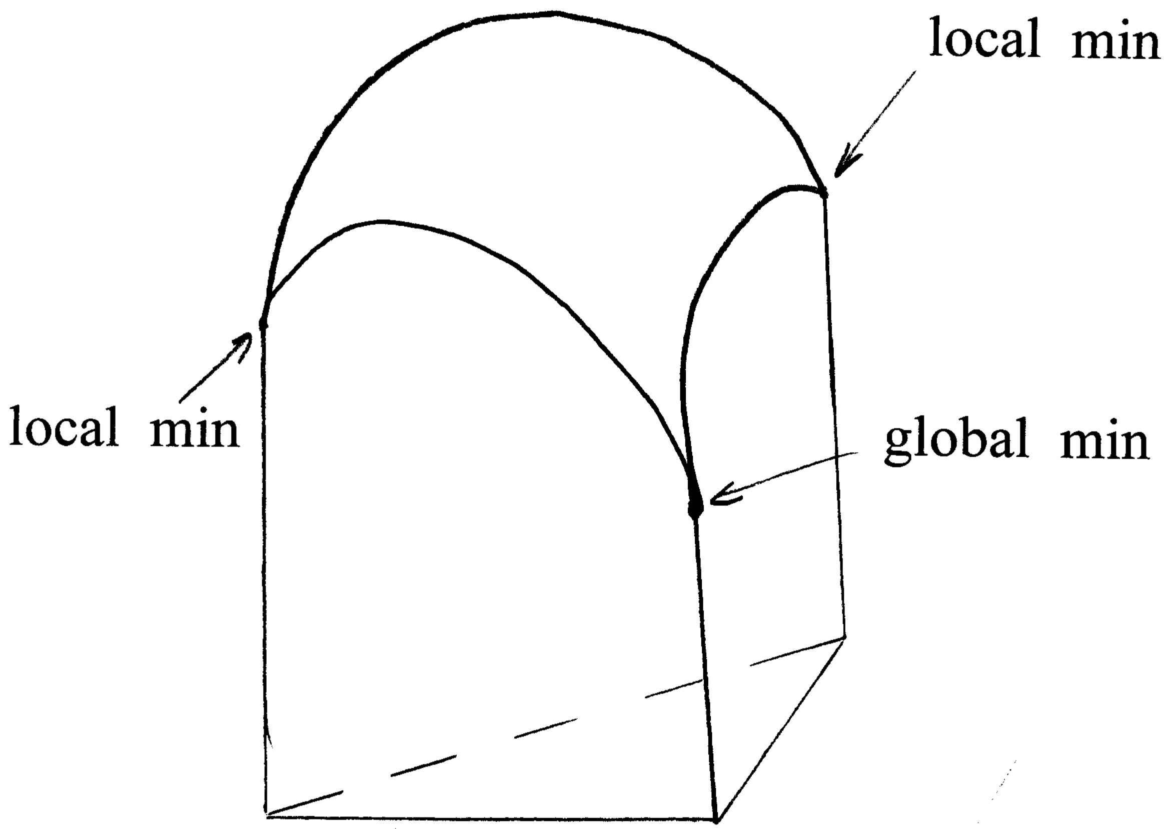
$$\forall \; x_1, \; x_2 \in \Omega \Rightarrow x = (1-\lambda)x_1 + \lambda x_2 \in \Omega$$

$$0 < \lambda < 1$$

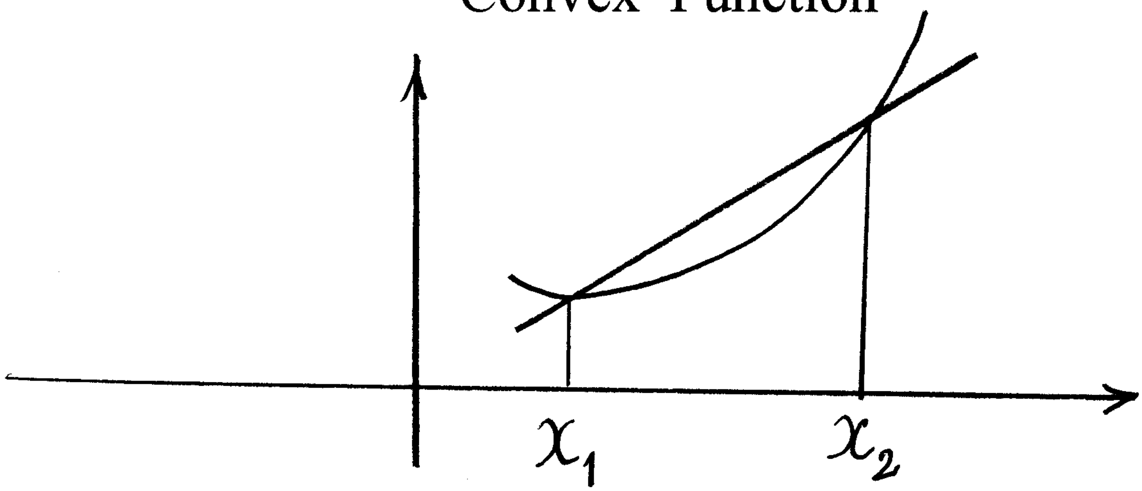
if Ω_i *is convex then*

$\Omega = \prod \Omega_i$ *is convex*




local min
local min
global min

Convex Function

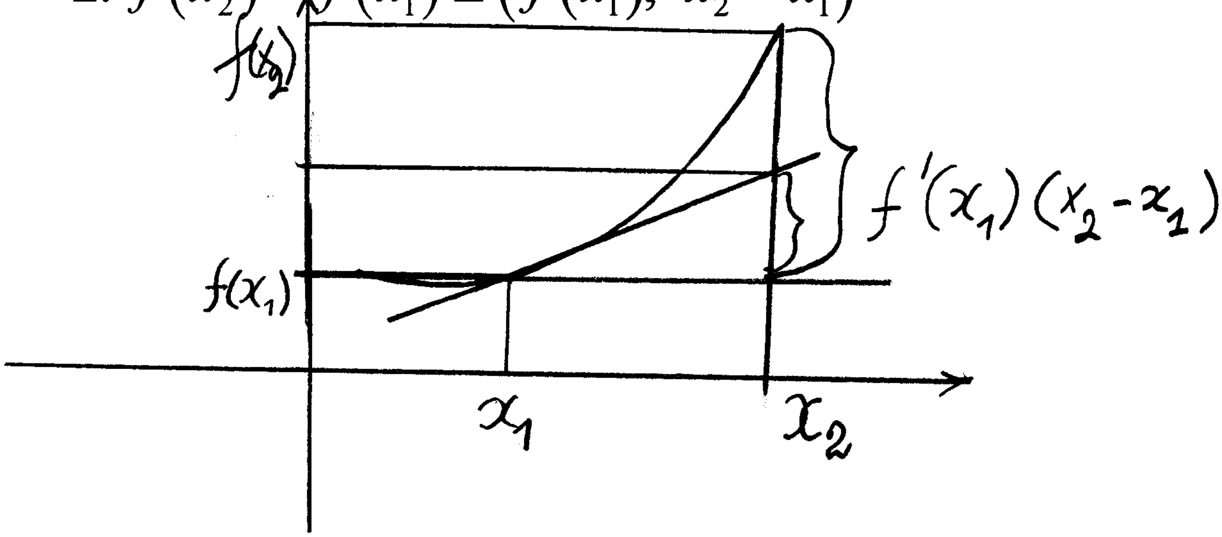


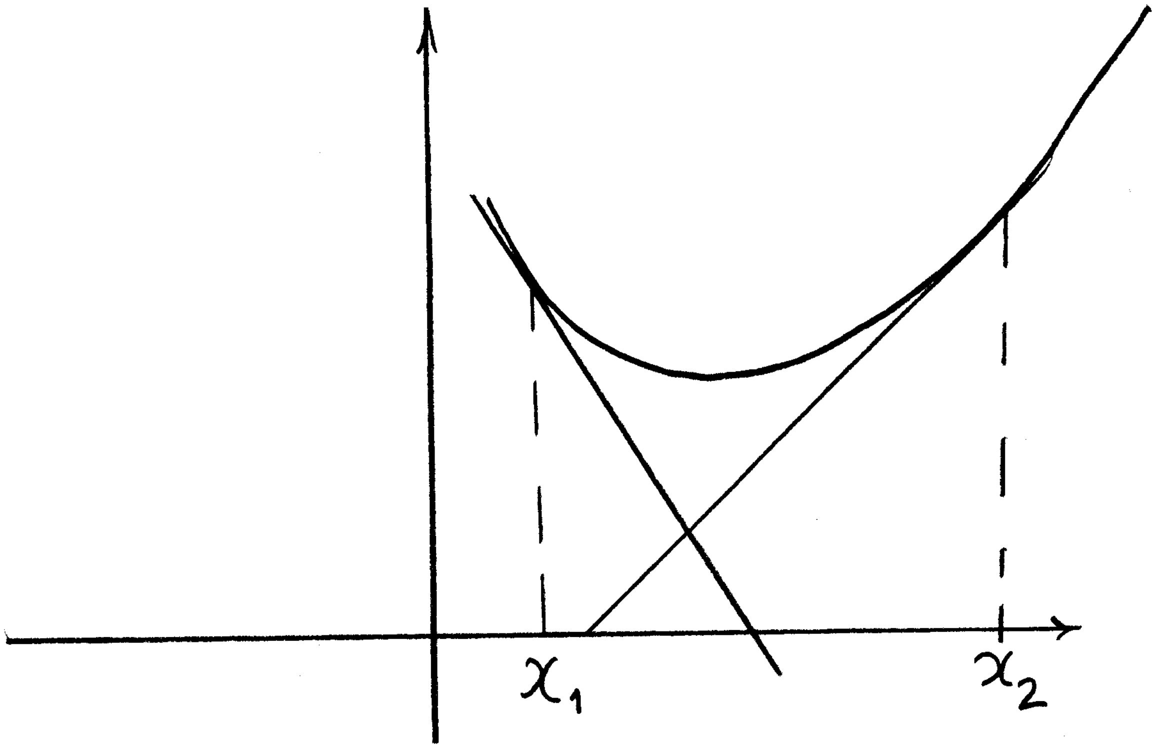
1. $f\left((1-\lambda)x_1+\lambda x_2\right)\le(1-\lambda)f(x_1)+\lambda f(x_2)$

Gradient

$$\nabla f(x)=\left(\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n}\right)$$

2. $f(x_2)-f(x_1)\ge\left(f(x_1),\ x_2-x_1\right)$





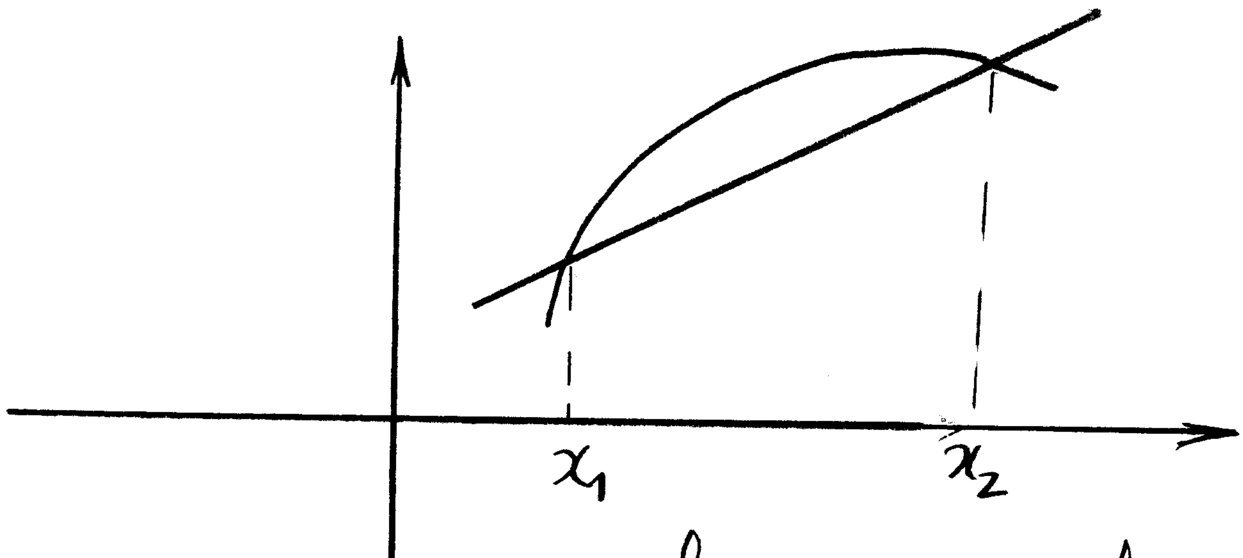
3. $\left(\nabla f(x_1) - \nabla f(x_2),\ x_1 - x_2\right) \geq 0$

Hessian

$$\nabla^2_{xx} f(x) = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1 \partial x_1}, \ldots, \dfrac{\partial^2 f}{\partial x_1 \partial x_n} \\ \dfrac{\partial^2 f}{\partial x_n \partial x_1}, \ldots, \dfrac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}$$

4. $\left(\nabla^2_{xx} f(x) y,\ y\right) \geq m(y, y)$

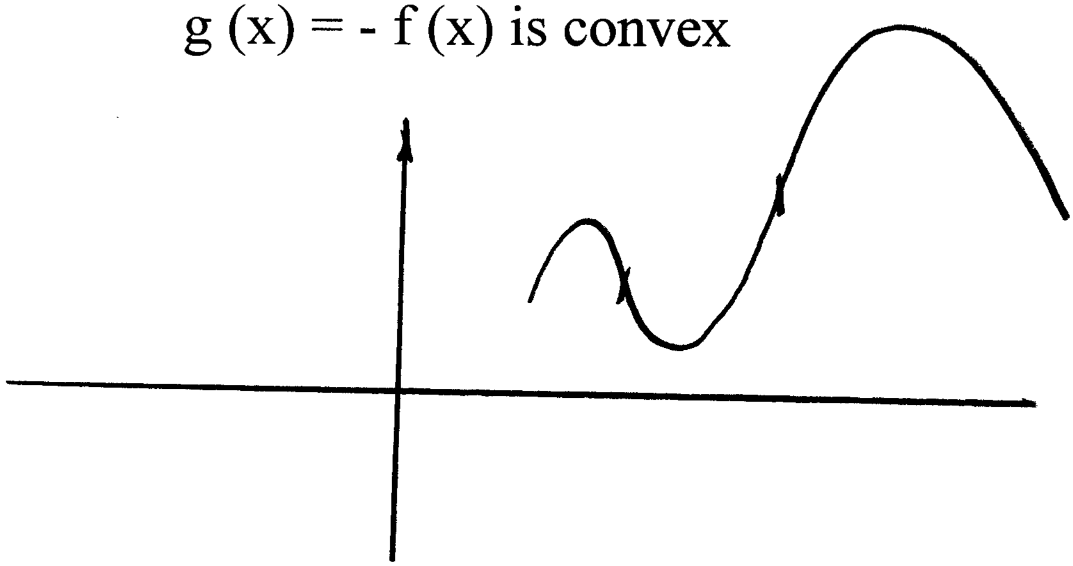
Concave Functions



$$f((1-\lambda)x_1+\lambda x_2) \geq (1-\lambda)f(x_1)+\lambda f(x_2)$$

f (x) is concave if

g (x) = - f (x) is convex



Neither convex nor concave

Linear Function

f (x) = ax + b
is both convex and concave

Examples

1) $f_1(x) = x^3$

2) $f_2(x) = \frac{1}{x}$

3) $f_3(x) = \ln x$

4) $f(x_1, x_2) = x_1^3 + 3x_1x_2 + x_2^2,$

$$\nabla f(x) = \begin{pmatrix} 3x_1^2 + 3x_2 \\ 3x_1 + 2x_2 \end{pmatrix}$$

$$H(x) = \begin{pmatrix} 6x_1 & 3 \\ 3 & 2 \end{pmatrix}$$

$x_1 > 0,$ $\quad 6x_1.2 - 9 \geq 0$

$$x_1 \geq \frac{3}{4}$$

1) If $f(x)$ is convex then

$$\Omega = \{x:\ f(x) \le c\} \text{ is convex for any } c$$

2) If $f_i(x)$ is convex and $\alpha_i \ge 0$ then

$$f(x) = \sum_{i=1}^{m} \alpha_i f_i(x) \text{ is convex}$$

3) If $y = f(x)$ concave and $f(x) > 0$ then

$$Z = \frac{1}{f(x)} \text{ is convex}$$

4) If $y = f(x) \ge 0$ and concave then

$$Z = \ln f(x) \text{ is concave}$$

Unconstrained Optimization

min f (x) = f (x*)

or

$$x^* = avg\ \min\left\{ f(x) /\ \ x \in R^h \right\}$$

Two basic methods

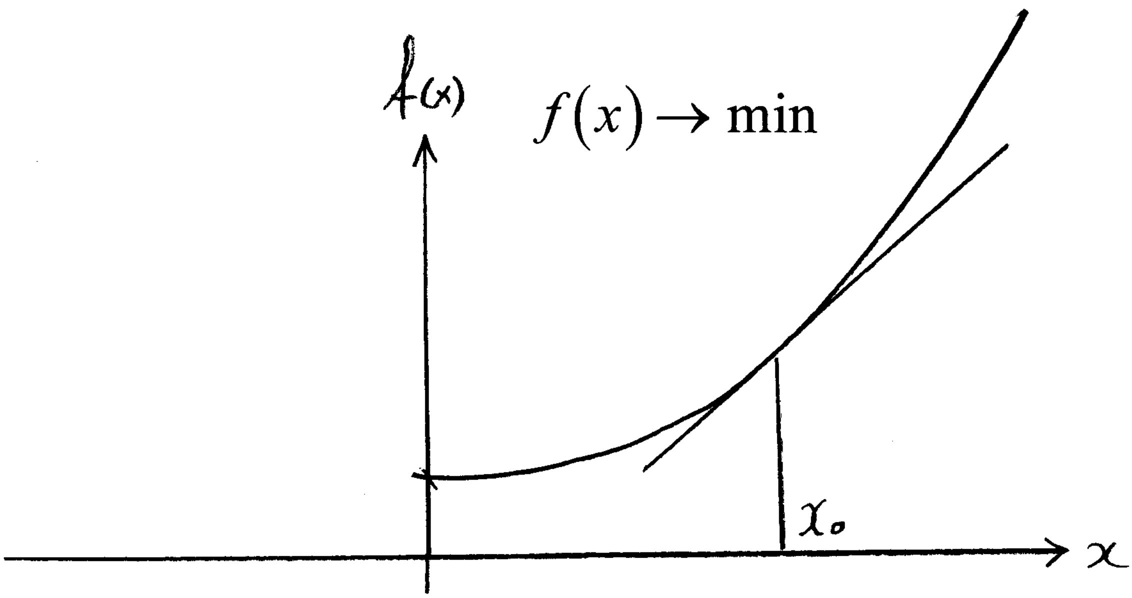
Gradient Method

$$x^{s+1} = x^s - t\nabla f\left(x^s\right), \qquad t > 0$$

Newton Method

$$x^{s+1} = x^s - \left(\nabla_{xx}^2 f\left(x^s\right)\right)^{-1} \ \nabla f\left(x^s\right)$$

Unconstrained minimization

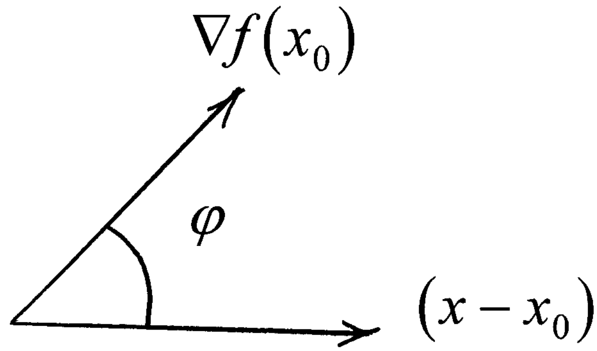


$$\tilde{f}(x) \approx f(x_0) + \nabla f^T(x_0)(x - x_0)$$

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

$$\min \ \tilde{f}(x)$$

$$\| x - x_0 \| \leq 1$$



$$\nabla f^{T}\left(x^{0}\right)\left(x-x^{0}\right)= \left\|\nabla f\left(x_{0}\right)\right\| \left\|x-x_{0}\right\| \cos\varphi$$

$$\varphi = 180^{0}$$

$$x - x^{0} = -\frac{\nabla f\left(x_{0}\right)}{\left\|\nabla f\left(x_{0}\right)\right\|}$$

$$x^{1} = x^{0} - t\,\frac{\nabla f\left(x_{0}\right)}{\left\|\nabla f\left(x_{0}\right)\right\|}$$

$$\min \ \tilde{f}(x)$$

$$x: \ \| x - x^0 \| \le 1 \qquad\qquad \| x \| = \left(x_1^2 + \ldots + x_n^2\right)^{\frac{1}{2}}$$

then we are dealing with the following problem

$$\min \left(\nabla f\left(x^0\right), \ \zeta\right)$$

$$\| \zeta \| = 1$$

$$\left(\nabla f\left(x^0\right), \ \zeta\right) = \left\| \nabla f\left(x^0\right) \right\| \ \| \zeta \| \cos \varphi$$

$$\zeta = - \frac{\nabla f\left(x^0\right)}{\left\| \nabla f\left(x^0\right) \right\|}$$

This is the steepest descent direction

Another way of looking on this matter

is the following

Let consider the function

$$f\left(x^0 + t\zeta\right)$$

$$\frac{d}{dt} f\left(x^0 + t\zeta\right) = \left(\nabla f\left(x^0 + t\zeta\right), \zeta\right) \ / \, t = 0$$

$$= \left(\nabla f\left(x^0\right), \zeta\right)$$

The value $\left(\nabla f\left(x^0\right), \zeta\right)$ *characterizes the velocity of the function increase or decrease in the direction* ζ
Again increase or decrease will depend on the angle between $\nabla f\left(x^0\right)$ *and* ζ

$$\min\left(\nabla f\left(x^0\right), \zeta\right) \Rightarrow \zeta = -\frac{\nabla f\left(x^0\right)}{\left\|\nabla f\left(x^0\right)\right\|}$$

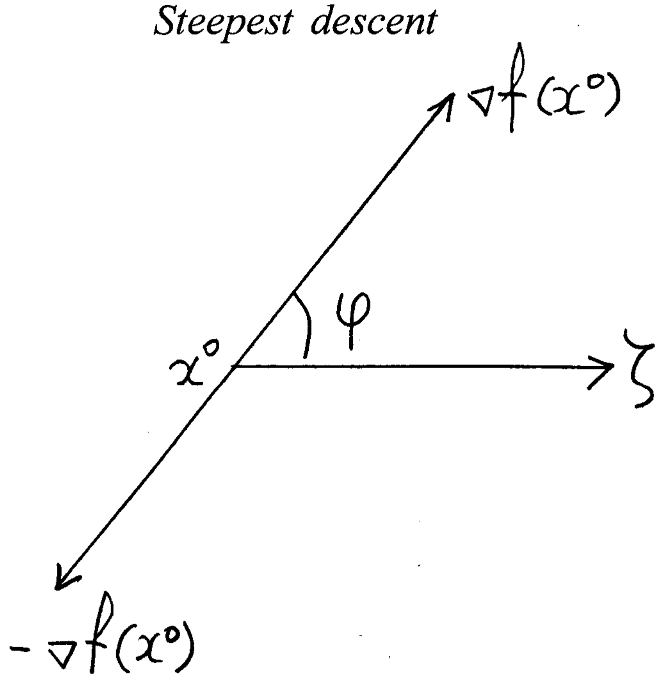
$$\|\zeta\| \leq 1$$

$$\alpha^{s+1} = x^s - t\nabla f\left(x^s\right)$$

$$t_{s+1} = avg \min_{t\geq 0} f\left(x^s - t\nabla f\left(x^s\right)\right)$$

$$= \left(\nabla f\left(x^s - t\nabla f\left(x^s\right)\right), \nabla f\left(x^s\right)\right) = 0$$

Steepest descent



$$\left(\nabla f(x)-\nabla f(y),\ x-y\right) \geq m\ \|x-y\ \|$$

$$\|\nabla f(x)-\nabla f(y)\ \| \leq L\ \|x-y\ \|$$

$$x^{s+1}=x^{s}-t\nabla f\left(x^{s}\right)$$

$$x^{s+1}-x^{*}=x^{s}-x^{*}-t\nabla f\left(x^{s}\right)$$

$$\left(x^{s+1}-x^{*},\ x^{s+1}-x^{*}\right)=\left(x^{s}-x^{*}-t\nabla f\left(x^{s}\right),\ x^{s}-x^{*}-t\nabla f\left(x^{s}\right)\right)$$

$$\begin{aligned}\|x^{s+1}-x^{*}\ \|^{2}=&\left(x^{s}-x^{*},\ x^{s}-x^{*}\right)\\ &-2t\left(x^{s}-x^{*},\ \nabla f\left(x^{s}\right)-\nabla f\left(x^{*}\right)\right)\\ &+t^{2}\ \|\nabla f\left(x^{s}\right)-\nabla f\left(x^{*}\right)\ \|^{2}\end{aligned}$$

$$=\ \|x^{s}-x^{*}\ \|^{2}-2tm\ \|x^{s}-x^{*}\ \|^{2}+t^{2}L^{2}\ \|x^{s}-x^{*}\ \|^{2}$$

$$=\ \|x^{s}-x^{*}\ \|\left(1-2tm+t^{2}L^{2}\right)$$

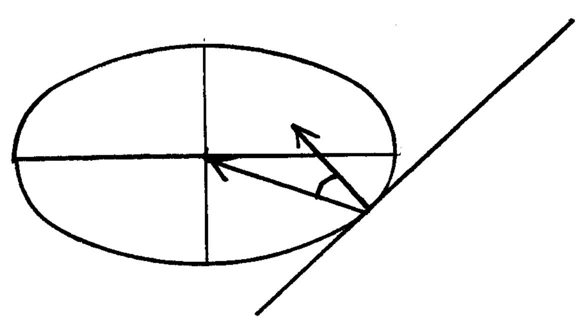
$$q(t) = 1 - 2tm + t^2 L^2 \to \min_{t}$$

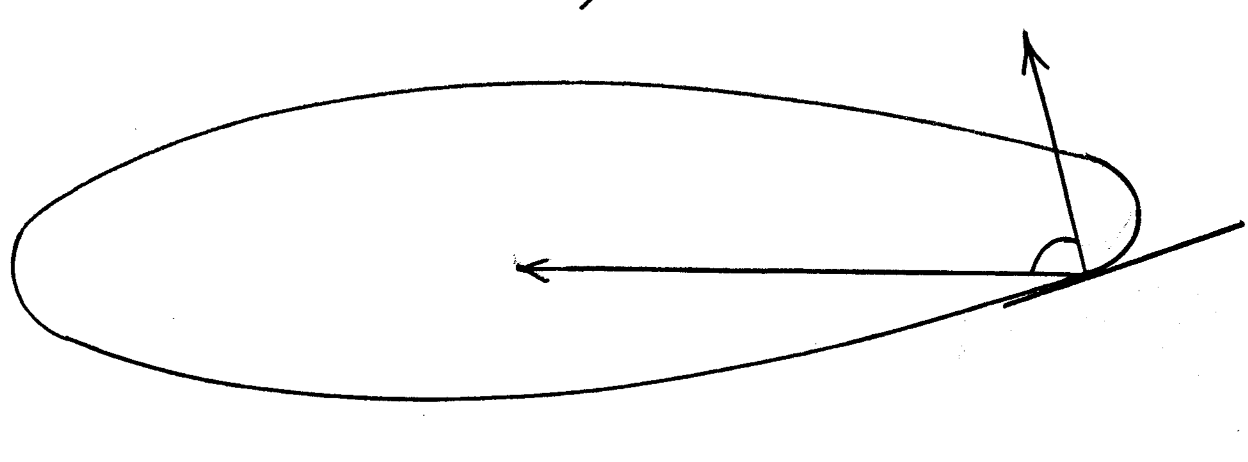
$$-2m + 2tL^2 = 0$$

$$t = \frac{m}{L^2}$$

$$q\left(\frac{m}{L^2}\right) = 1 - \frac{m}{L^2}$$

$$\left\| \alpha^{s+1} - x^* \right\| \le \sqrt{1 - \frac{m}{L^2}} \cdot \left\| x^s - x^* \right\|$$





Cauchy 1847

Steepest descent

$$t_{s+1} : \min_t f\left(x^s - t\nabla f\left(x^s\right)\right)$$

$$x^{s+1} = x^s - t_{s+1}\nabla f\left(x^s\right)$$

$$q = \sqrt{\frac{L-m}{L+m}} = \sqrt{\frac{1-\sigma}{1+\sigma}}$$

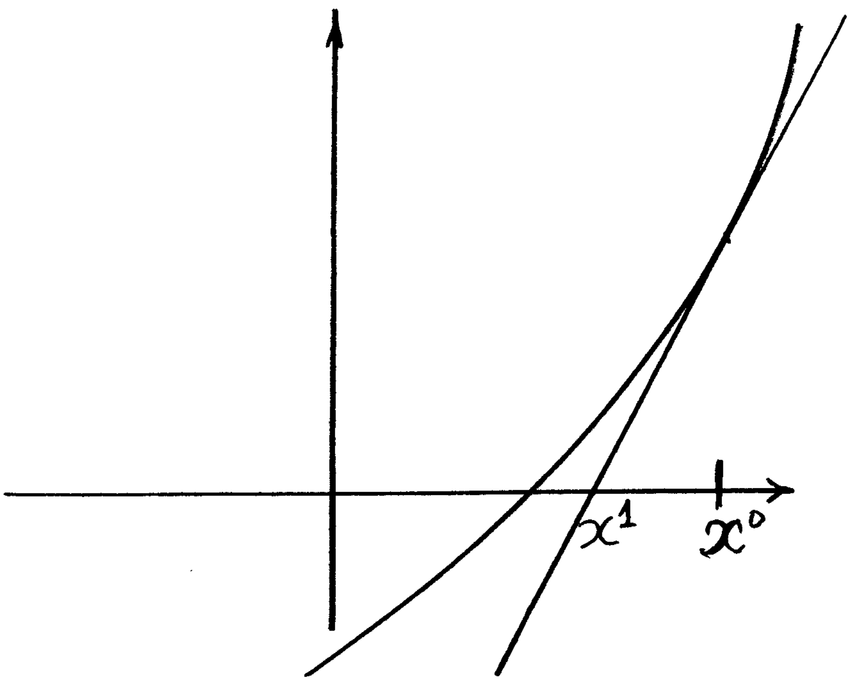
$\sigma = \frac{m}{L}$ $-$ *condition number of the unconstr.*

If $\sigma > 0$ *is small then*

the unconstr. optimization problem is

ill – condtioned

Newton Method 1660



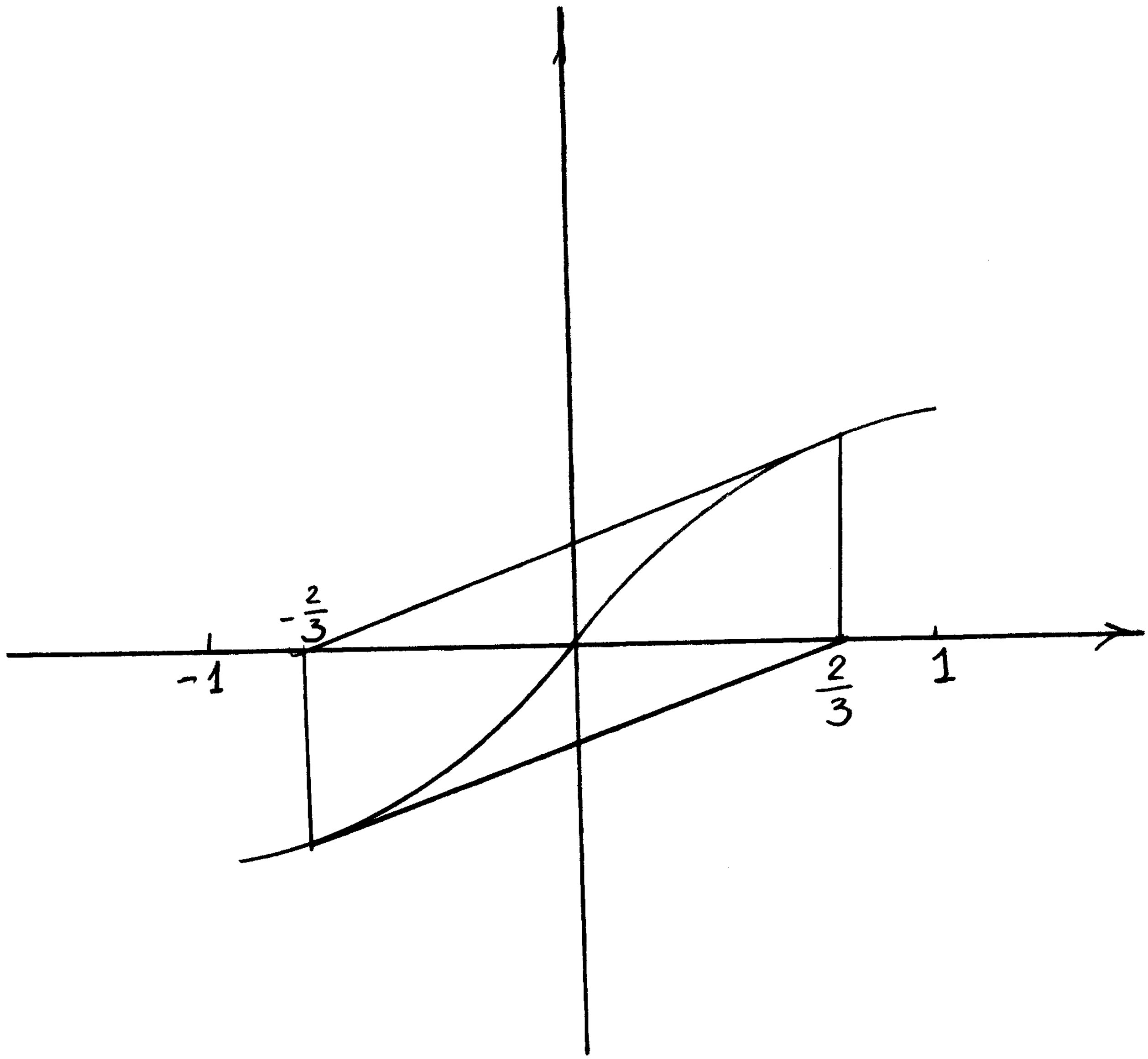
$g(x) = 0$

$$0 = g(x_0 + \Delta x) = g(x_0) + g'(x_0)(x - x_0)$$

$$x - x_0 = -(g'(x_0))^{-1} g(x_0)$$

$$x_1 = x_0 - (g'(x_0))^{-1} g(x_0)$$

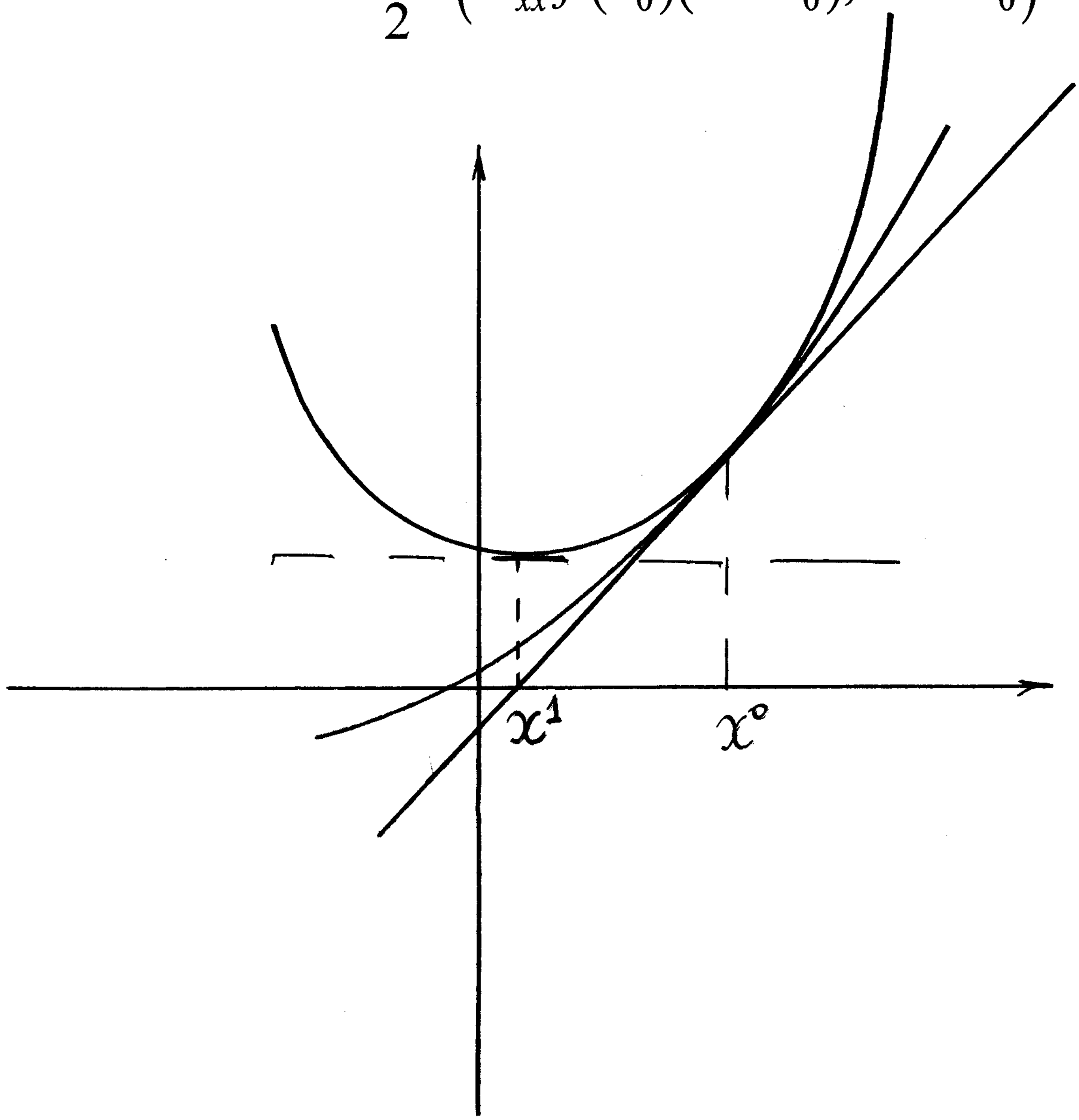
$$x_{s+1} = x_s - (g'(x_s))^{-1} g(x_s)$$



$$g(x)=\begin{cases}-(x-1)^2+1\ , & 0\le x\le 1\\ (x+1)^2-1\ , & -1\le x\le 0\end{cases}$$

$$f(x) \to \min \Leftrightarrow \nabla f(x) = g(x) = 0$$

$$\tilde{f}(x) = f(x_0) + (\nabla f(x_0),\ x - x_0)$$
$$+ \frac{1}{2}\ (\nabla^2_{xx} f(x_0)(x - x_0),\ \ x - x_0)$$



$$\min \ \tilde{f}(x) \Rightarrow \nabla \tilde{f}(x) = 0$$

$$\nabla \tilde{f}(x) = \nabla f(x_0) + \nabla^2_{xx} f(x_0)(x - x_0)$$

$$x - x_0 = -\left(\nabla^2_{xx} f(x_0)\right)^{-1} \nabla f(x_0)$$

$$x_1 = x_0 - \left(\nabla^2_{xx} f(x_0)\right)^{-1} \nabla f(x_0)$$

or

$$x_{s+1} = x_s - \left(\nabla^2_{xx} f(x_s)\right)^{-1} \nabla f(x_s)$$

$$\left\| x^{s+1} - x^* \right\| \leq c \left\| x^s - x^* \right\|^2$$

$$x^{s+1} = x^s - \nabla^2 f^{-1}\left(x^s\right) \nabla f\left(x^s\right)$$

$$x^{s+1} - x^* = x^s - x^* - \nabla^2 f^{-1}\left(x^s\right) \nabla f\left(x^s\right)$$

$$\left\| x^{s+1} - x^* \right\|^2 = \left(x^{s+1} - x^*,\ x^s - x^* - \nabla^2 f^{-1}\left(x^s\right) \nabla f\left(x^s\right)\right)$$

$$\leq \left(x^{s+1} - x^*,\ x^s - x^* - \nabla^2 f^{-1}\left(x^s\right)\left(\nabla f\left(x^s\right) - \nabla f\left(x^*\right)\right)\right)$$

$$= \left(x^{s+1} - x^*,\ \nabla^2 f^{-1}\left(x^s\right)\left(\nabla^2 f\left(x^s\right)\left(x^s - x^*\right) - \nabla f\left(x^s\right) - \nabla f\left(x^*\right) \right)\right)$$

$$= \left(x^{s+1} - x^*,\ \nabla^2 f^{-1}\left(x^s\right)\left(\nabla^2 f\left(x^s\right)\left(x^s - x^*\right) - \nabla^2 f\left(x^s + \theta\left(x^* - x^s\right)\right)\left(x^s - x^*\right) \right)\right)$$

$$= \left(x^{s+1} - x^*,\ \nabla^2 f^{-1}\left(x^s\right)\left(\nabla^2 f\left(x^s\right) - \nabla^2 f\left(x^s + \theta\left(x^* - x^s\right)\right)\right)\left(x^s - x^*\right) \right)$$

$$\left(\nabla^2 f(x)\zeta,\ \zeta\right) \geq m(\zeta,\ \zeta)$$

$$\left\| \nabla^2 f(x) - \nabla^2 f(y) \right\| \leq L \ \| x - y \|$$

$$\| x^{s+1} - x^* \|^2$$

$$\leq \ \left\| \nabla^2 f^{-1}(x^s) \right\| \ \left\| \nabla^2 f(x^s) - \nabla^2 f\left(x^s + \theta(x^* - x^s)\right) \right\|$$

$$\| x^{s+1} - x^* \| \ \| x^s - x^* \|$$

$$\| x^{s+1} - x^* \| \leq m^{-1} L \ \left\| x^s - \left(x^s - \theta(x^* - x^s)\right) \right\| \ \| x^s - x^* \|$$

$$= m^{-1} L \theta \ \| x^s - x^* \|^2$$

$$\| x^{s+1} - x^* \| \leq c \ \| x^s - x^* \|^2$$

$$c = m^{-1} L \theta$$