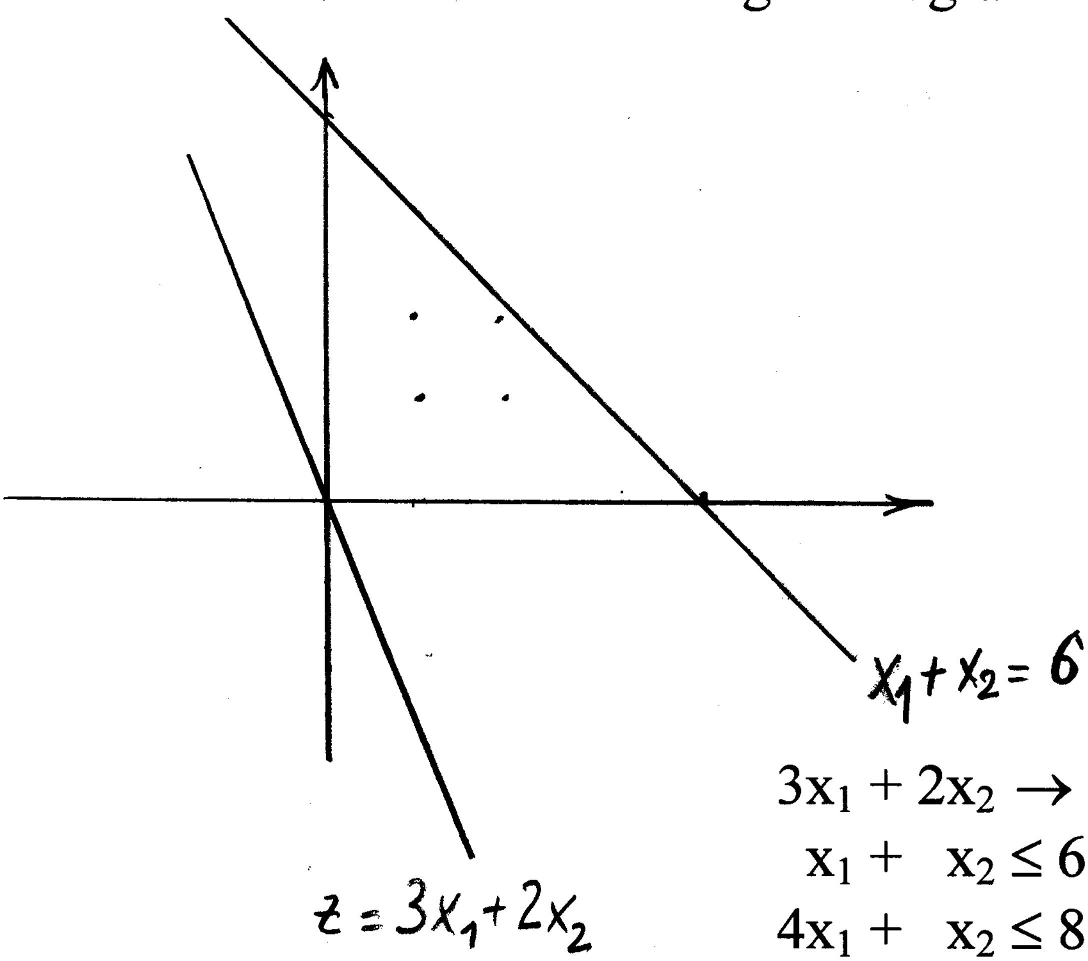
Integer Programming

$$\begin{aligned} \max\ & 3x_1 + 2x_2 \\ & x_1 + x_2 \le 6 \\ & x_1 \ge 0,\ x_2 \ge 0 \end{aligned}$$

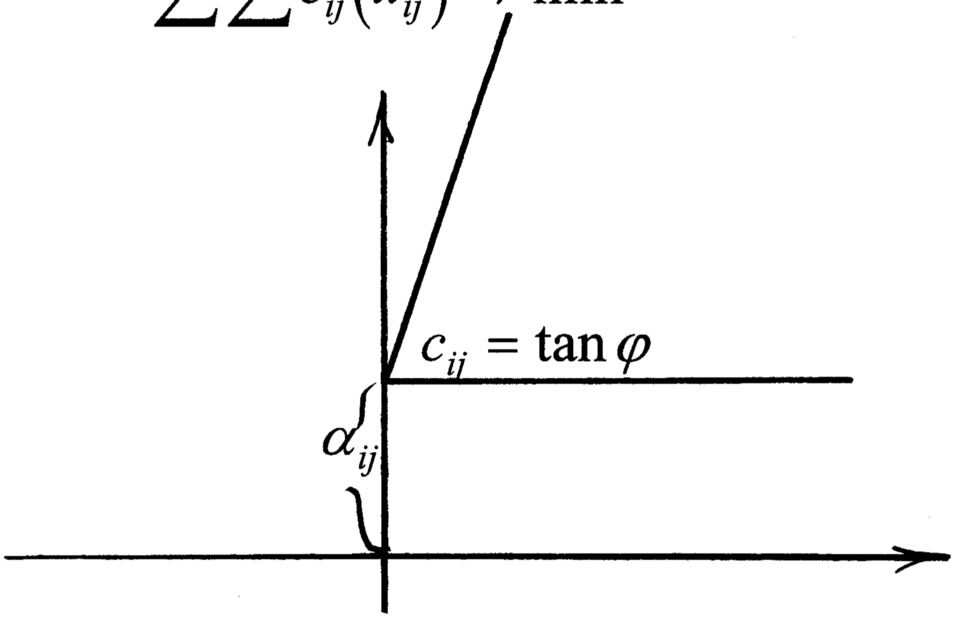
LP relaxation for Integer Programming



$$\begin{aligned} & 3x_1 + 2x_2 \to \max \\ & x_1 + x_2 \le 6 \\ & 4x_1 + x_2 \le 8 \\ & x_1 \ge 0,\ x_2 \ge 0 \end{aligned}$$

Fixed Charge Transportation

$$\sum\sum c_{ij}(x_{ij}) \rightarrow \min$$



$$c_{ij}(x_{ij}) = \begin{cases} c_{ij}x_{ij} + \alpha_{ij}, & x_{ij} > 0 \\ 0 \quad , & x_{ij} = 0 \end{cases}$$

$$m_{ij} = \min\{a_i,\ b_j\}$$

$$\sum\sum c_{ij}x_{ij} + \sum\sum \alpha_{ij}y_{ij} \to \min$$

$$\sum_j x_{ij} = a_i, \quad \sum_i x_{ij} = b_i$$

$$0 \le x_{ij} \le m_{ij}y_{ij}$$

$$y_{ij} \in \{0,\ 1\}$$

	Shirts	Shorts	Pants	
Labor	3	2	6	150
Cloth	4	3	4	100

	Shirts	Shorts	Pants
Sales Pr.	$12	$8	$15
var. Cost	$6	$4	$8

Machinery I - $200, II - $150, III - $100

Shirts – x_1, *Shorts* – x_2, *Pants* – x_3

Mach I – y_1, *Mach II* – y_2 *Mach III* – y_3

$$z = 6x_1 + 4x_2 + 7x_3 - 200y_1 - 150y_2 - 100y_3$$

$$3x_1 + 2x_2 + 6x_3 \leq 150$$

$$4x_1 + 3x_2 + 4x_3 \leq 100$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$$

$$y_i \in \{0,\ 1\}$$

$$x_1 \leq \mu_1 y_1$$

$$x_2 \leq \mu_2 y_2$$

$$x_3 \leq \mu_3 y_3$$

Either or Constraints

Dorian Auto is considering manufacturing three types of autos: compact, midsize & large

6000 tons of steel

60000 hrs. of labor is available

$$\text{Constraints} \quad x_1 \le 0 \quad or \quad x_1 \ge 1000$$
$$x_2 \le 0 \quad or \quad x_2 \ge 1000$$
$$x_3 \le 0 \quad or \quad x_3 \ge 1000$$

	Comp.	Midsize	Large
Steel	1.5T	3T	5T
Labor	30hr	25hr	40hr
Profit yielded	\$2000	\$3000	\$4000

$x_1 - comp.\ cars$ $\qquad$ $x_1 \le 2000$

$x_2 - midsize$ $\qquad$ $x_2 \le 2000$

$x_3 - large$ $\qquad$ $x_3 \le 1200$

$$x_1 \le 2000 y_1$$

$$1000 - x_1 \le 2000(1 - y_1) \qquad y_1 \in \{0, 1\}$$

$$x_2 \le 2000 y_2$$

$$1000 - x_2 \le 2000(1 - y_2) \qquad y_2 \in \{0, 1\}$$

$$x_3 \le 1200 y_3$$

$$1000 - x_3 \le 1200(1 - y_3)$$

$$1.5x_1 + 3x_2 + 5x_3 \le 6000$$
$$30x_1 + 25x_2 + 40x_3 \le 60000$$
$$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0 \quad x_i - \text{integer}$$
$$y_i \in \{0,\ 1\}$$

$$z = 2x_1 + 3x_2 + 4x_3$$
$$x_1 \le 2000y_1$$
$$1000 - x_1 \le 2000(1 - y_1)$$
$$x_2 \le 2000y_2$$
$$1000 - x_2 \le 2000(1 - y_2)$$
$$x_3 \le 1200y_3$$
$$1000 - x_3 \le 1200(1 - y_3)$$
$$1.5x_1 + 3x_2 + 5x_3 \le 6000$$
$$30x_1 + 25x_2 + 40x_3 \le 60000$$
$$x_i \ge 0, \quad y_i \in \{0,\ 1\}$$

If - Then Constraints

If $f(x_1,\ldots, x_n) > 0$ then $g(x_1,\ldots, x_n) \leq 0$ must be satisfied

$$-g \leq \mu y$$
$$f \leq \mu(1-y)$$

If $f > 0$ then only $y = 0$ is possible, so

$$-g \leq 0 \quad \text{or} \quad g \geq 0$$

Set Covering

		*Math*1	*Math*2	*OR*1	*OR*2	*CS*1	*CS*2
x_1	*Calculus*	1	1				
x_2	*OR*	1	1	1	1	1	
x_3	*Data Struct*	1				1	1
x_4	*Business Stat*	1	1	1			
x_5	*Comp. Simul*				1	1	
x_6	*Introd. to CP*						1
x_7	*Forecasting*	1		1	1		

Cal $\Rightarrow$ *Bus. Stat.*

Intr. $\Rightarrow$ *Comp. Simul*

Intr. $\Rightarrow$ *Business Stat.*

Bus. Stat $\Rightarrow$ *Forecast*

*Math*1 – 2
*Math*2 – 2
*OR*1 – 1
*OR*2 – 2
*CS*1 – 1
*CS*2 – 1

$$\sum x_i \to \min$$

$$x_1 + x_2 + x_3 + x_4 + x_7 \geq 2$$

$x_1 \geq x_4$ $\qquad x_1 + x_2 + x_4 \geq 1$

$x_6 \geq x_5$ $\qquad x_2 + x_4 + x_7 \geq 1$

$x_6 \geq x_4$ $\qquad x_2 + x_5 + x_7 \geq 2$

$x_4 \geq x_7$ $\qquad x_2 + x_3 + x_5 \geq 1$

$$x_3 + x_6 \geq 1, \qquad x_i \in \{0,\ 1\}$$

	Math1	Math2	OR1	OR2	CS1	CS2	
Calculus	1	1					x_1
OR	1	1	1	1	1		x_2
Data Struct	1				1	1	x_3
Bus. Stat	1	1	1				x_4
Comp. Simul				1	1	1	x_5
Int. CP					1		x_6
Forecasting	1		1	1			x_7
	1	2	2	1	2	1	

$$\sum x_i \to \min$$

$$\begin{aligned}
x_1 + x_2 + x_3 + x_4 + x_7 &\geq 1 \\
x_1 + x_2 + x_4 &\geq 2 \\
x_2 + x_4 + x_7 &\geq 2 \\
x_2 + x_5 + x_7 &\geq 1 \\
x_2 + x_3 + x_5 + x_6 &\geq 2 \\
x_3 + x_5 &\geq 1 \\
x_i \in \{0,\ 1\}
\end{aligned}$$

Set Covering

0	10	20	30	30	20
10	0	25	35	20	10
20	25	0	15	30	20
30	35	15	0	15	25
30	20	30	15	0	14
20	10	20	25	14	0

x_1	1	1				
x_2	1	1				1
x_3			1	1		
x_4			1	1	1	
x_5				1	1	1
x_6		1			1	1

$$z = x_1 + x_2 + \ldots + x_6$$

$$x_3 + x_4 + x_5 \geq 1 \qquad x_1 + x_2 \geq 1$$

$$x_4 + x_5 + x_6 \geq 1 \qquad x_1 + x_2 + x_6 \geq 1$$

$$x_2 + x_5 + x_6 \geq 1 \qquad x_3 + x_4 \geq 1$$

$$x_i \in \{0,\ 1\}$$

Knapsack Problem

Branch & Bound

$$5x_1 + 8x_2 + 3x_3 + 7x_4 \to \max$$
$$3x_1 + 5x_2 + 2x_3 + 4x_4 \le 10$$
$$x_i \in \{0,\ 1\}$$

$$\frac{7}{4} > \frac{5}{3} > \frac{8}{5} > \frac{3}{2}$$

$$7x_1 + 5x_2 + 8x_3 + 3x_4 \to \max$$
$$4x_1 + 3x_2 + 5x_3 + 2x_4 \le 10$$
$$x_i \in \{0,\ 1\}$$

Relaxation LP

$$7x_1 + 5x_2 + 8x_3 + 3x_4 \to \max$$

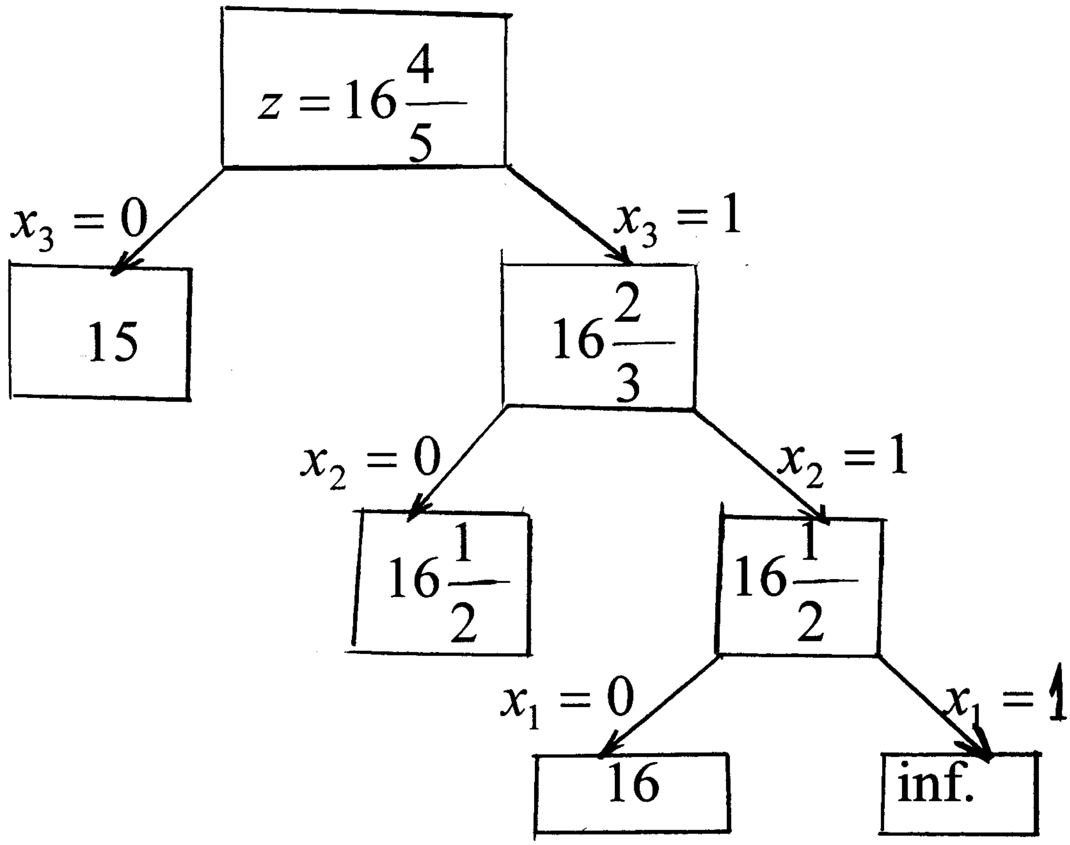
$$4x_1 + 3x_2 + 5x_3 + 2x_4 \le 10$$

$$0 \le x_i \le 1$$

Solution:

$$x_1 = 1,\ x_2 = 1,\ x_3 = \frac{3}{5}$$

$$z = 7 + 5 + \frac{3.8}{5} = 16\frac{4}{5} \quad \textit{upper bound}$$

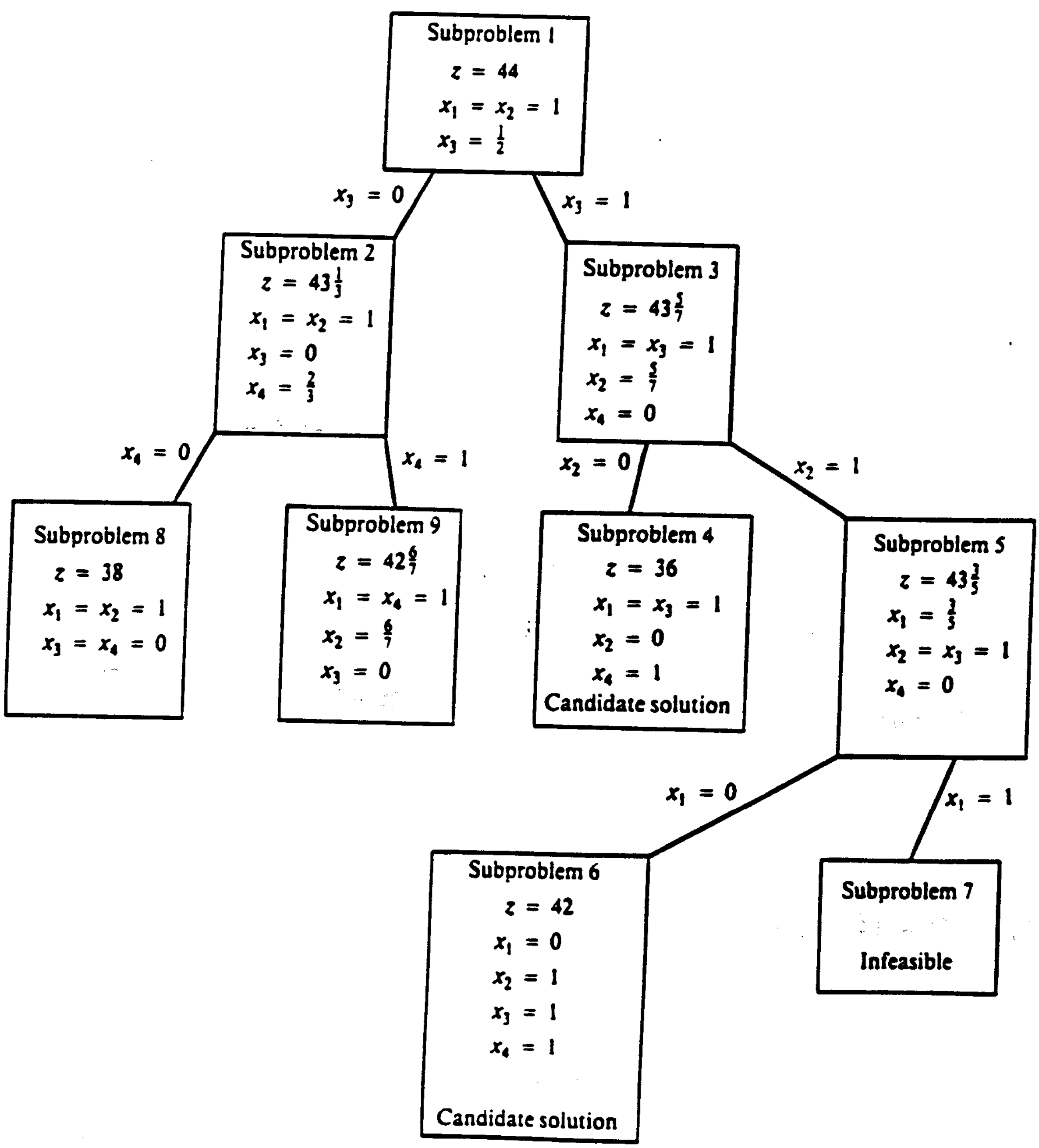


$z^* = 16, \quad x_1^* = 0, \quad x_2^* = x_3^* = x_4^* = 1$

$$16x_1 + 22x_2 + 12x_3 + 8x_4 \to \max$$

$$5x_1 + 7x_2 + 4x_3 + 3x_4 \le 14$$

$$x_i \in \{0,\ 1\}$$

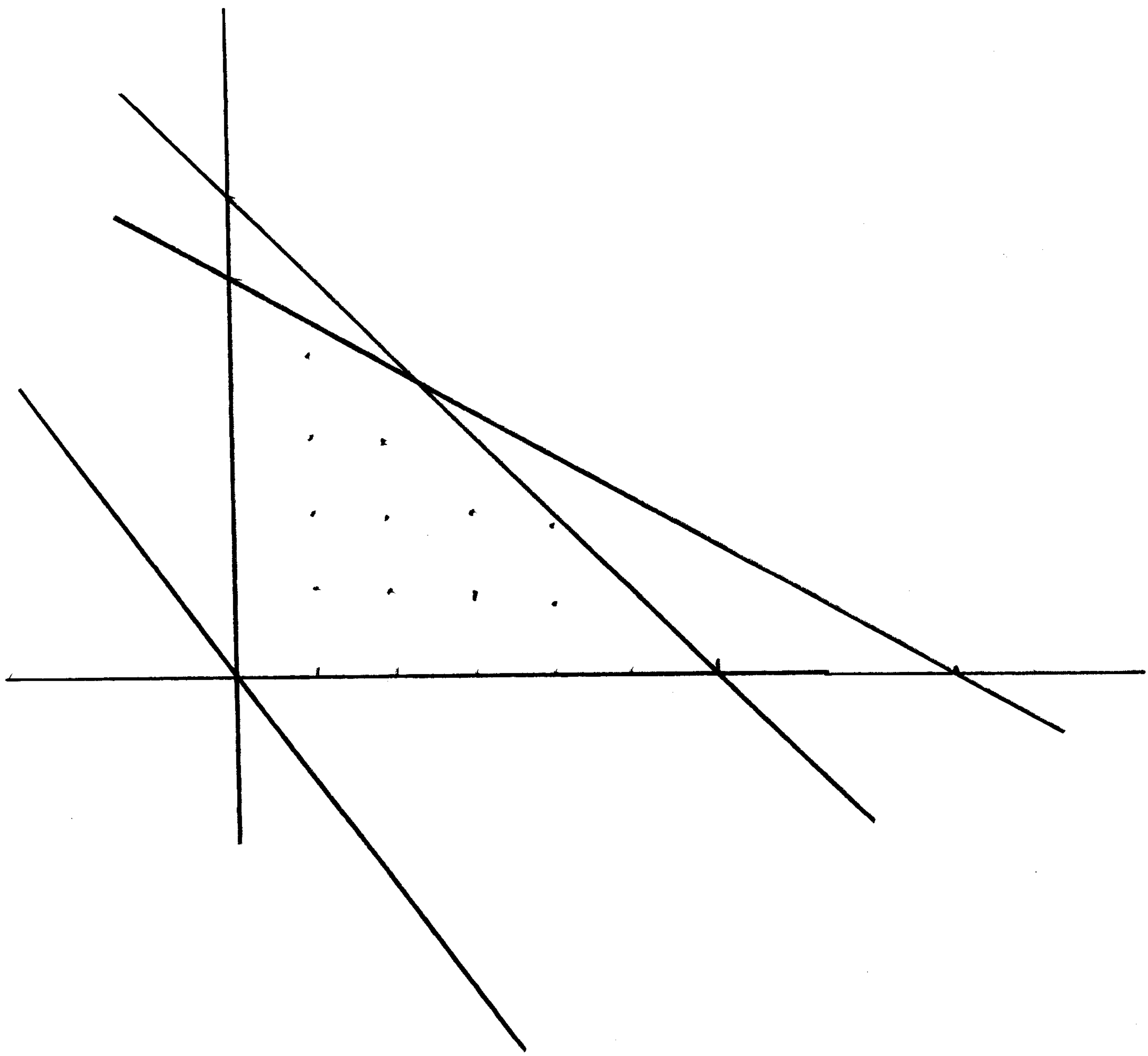


$$z = 8x_1 + 5x_2 \to \max$$

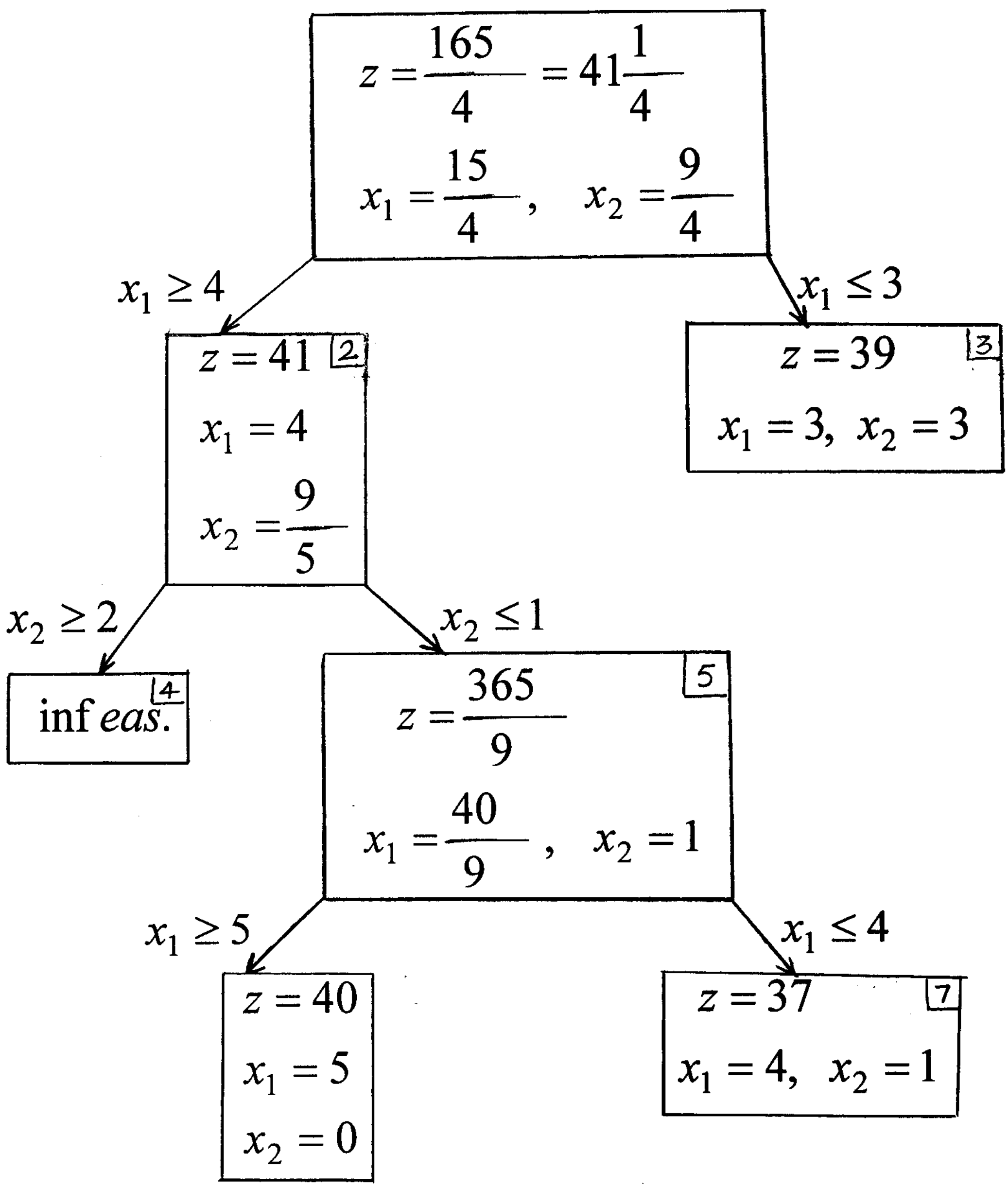
$$x_1 + x_2 \le 6$$

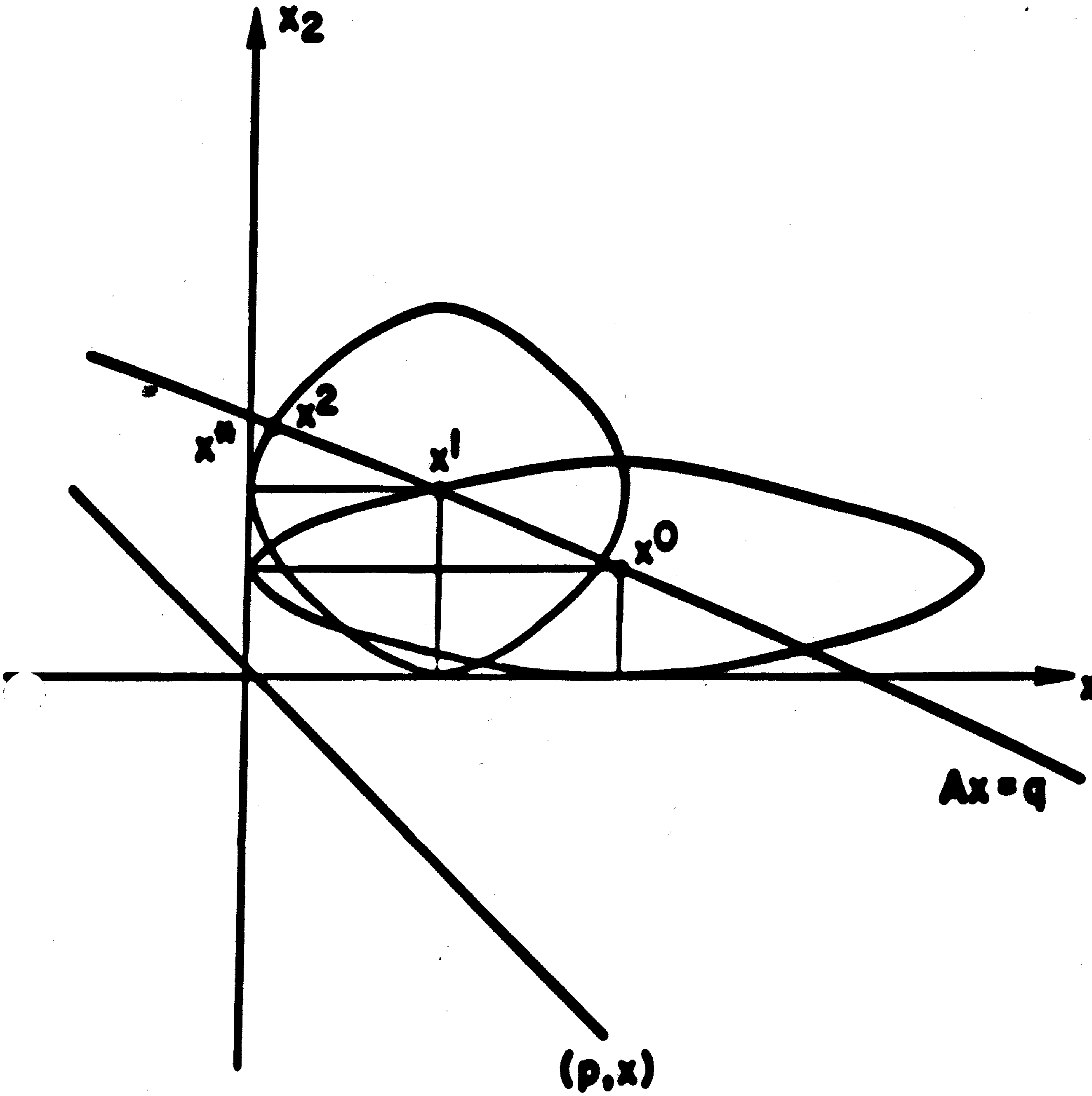
$$9x_1 + 5x_2 \le 45$$

$$x_1 \ge 0,\ x_2 \ge 0;\ x_i - \text{integer}$$



$$z = \frac{165}{4}, \quad x_1 = \frac{15}{4}, \quad x_2 = \frac{9}{4}$$





$$(p, x^{s+1} - x^*) \le \left(1 - \frac{\alpha}{\sqrt{n - m + \varepsilon_s}}\right)(p, x^s - x^*), \quad \varepsilon_s \to 0$$

I.I. Dikin 1967